Evolution Equations and Applications

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Epigraph

"The creative principle of Science resides in Mathematics. "
Albert Einstein.

"If you don't feel it, you won't be able to grasp it."
Johann Wolfgang von Goethe

Preface

This project concerns Evolution Equations in Banach spaces and lies at the interface between Functional Analysis, Dynamical Systems, Modeling Theory and Natural Sciences.

Evolution Equations include Partial Differential Equations (PDEs) with time t as one of the independent variables and arise from many fields of Mathematics as well as Physics, Mechanics and Material Sciences (e.g., Systems of Conservation Law from Dynamics, Navier-Stokes and Euler equations from Fluid Mechanics, Diffusion equations from Heat transfer and Natural Sciences, Klein-Gordon and Schrödinger equations from Quantum Mechanics, Cahn-Hilliard equations and Porous media equations from Material Sciences, Evolution equations with memory from Pharmacokinetics).

In this project, we present the fundamental theory of abstract Evolution Equations by using the semigroup approach (which arises naturally from well-posed Cauchy problems: Theorem 2.2.6) and Fixed-point methods. More precisely, first we review the basic notions of Functional Analysis and Differential Analysis, secondly we study the theory of semigroups of bounded linear operators, and thirdly we consider Linear Evolution Equations (with emphasis on the difference between the finite dimensional and the infinite dimensional case, that is due to domain restrictions) and moreover we give existence results (in appropriate sense) for Semilinear Evolution Equations of the form

$$\frac{du}{dt} = Au + f(t,u), \quad t > 0 \quad ; \quad u(0) = u_0$$

where A is a linear operator that generates a $\mathcal{C}_0$-semigroup and f satisfies certain conditions. As applications we start with the evolution equation $\partial_t u + \partial_x u = 0$ in $\mathbb{R}$ and then after we show the existence of solutions to some Homogeneous Heat Equations, classical Wave equations, nonlinear Heat Equation, and to some nonlinear Wave equation.

Acknowledgement

One may have pleasure in doing Mathematics, but:
It takes more than just a piece of paper, a pencil, the will and capability to recognize and assemble implications and equivalences to write a correct mathematical proof. And it is so unfortunate that sometimes a substantial part of what one may need has neither a material nor a Mathematical equivalent.
I am therefore by this, very indebted to those who gave me the strength and the extrinsic motivation to withstand the social and psychological adversities which have been persistent companions over the years.
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Dedication

To my **Lord and Saviour Jesus Christ** for His Everlasting Grace without which I would not have been here. Thank you Lord.

To my elder brother and his wife **Mr and Mrs Mbemboung** who took me under their care and taught me how to love and appreciate learning, for their unconditional and financial supports for the past years.

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Contents

CHAPTER 1

PRELIMINARIES

1.1 Basic notions of Functional Analysis

In this section we recall some definitions and results from linear functional analysis

Definition 1.1.1 *Let X be a linear space over a field $\mathbb{K}$, where $\mathbb{K}$ holds either for $\mathbb{R}$ or $\mathbb{C}$. A mapping $\|.\|$: $X \longrightarrow \mathbb{R}$ is called a norm provided that the following conditions hold:*

i) $\|x\| \geq 0$ *for all* $x \in X$, *and* $\|x\| = 0 \Leftrightarrow x = 0$

ii) $\|\lambda x\| = |\lambda| \|x\|$, *for all* $\lambda \in \mathbb{K}$, $x \in X$

iii) $\|x + y\| \leq \|x\| + \|y\|$, *for arbitrary* $x, y \in X$.

If X is a linear space and $\|.\|$ is a norm on X, then the pair $(X, \|.\|)$ is called a normed linear space over $\mathbb{K}$.

Should no ambiguity arise about the norm, we simply abbreviate this pair by saying that $\mathbf{X}$ is a normed linear space over $\mathbb{K}$.

Example . Let $X = \mathcal{C}([0,1])$ be the space of all real-valued continuous functions on $[0,1]$. Each of the following expressions defines on the vector space $\mathcal{C}([0,1])$ a norm which is in common use.

$$\|f\|_p = \left(\int_0^1 (|f(t)|)^p dt\right)^{\frac{1}{p}}, \text{ for every } f \in \mathcal{C}([0,1]), \text{ and any } p \in [1, \infty)$$
$$\|f\|_\infty = ess\,sup\,|f| = inf\{M \geq 0 : |f(x)| \leq M \quad a.e\}$$

Definition 1.1.2 ***(Equivalent norms)***
Two norms $\|.\|_1$ and $\|.\|_2$ defined on a linear space X are said to be equivalent if there exists $\alpha > 0$ and $\beta > 0$ constants such that

$$\alpha \|x\|_1 \leq \|x\|_2 \leq \beta \|x\|_1, \quad \forall x \in X.$$

Theorem 1.1.1 *In a finite dimensional linear space, all the norms are equivalent.*

Definition 1.1.3 *Every normed linear space E is canonically endowed with a metric d defined on $E \times E$ by*

$$d(x, y) \;=\; ||x - y|| \quad \forall\, x, y \in E.$$

Definition 1.1.4 ***(Cauchy sequence)***
A sequence $(x_n)_{n\geq 1}$ of elements of a normed vector space X is a Cauchy sequence if

$$\lim_{n,m\to\infty} \|x_n - x_m\| = 0.$$

That is, for any $\epsilon > 0$ there is an interger $N = N(\epsilon)$ such that $\|x_n - x_m\| < \epsilon$ whenever $n \geq N$ and $m \geq N$.

Remark. In a normed linear space, every Cauchy sequence $(x_n)_{n\geq 1}$ is bounded; i.e, there exists a constant $M \geq 0$ such that $||x_n|| \leq M$, $\forall n \geq 1$. (See also Definition 1.1.5 below)

Definition 1.1.5 ***(convergent sequence)***
A sequence $(x_n)_{n\geq 1}$ of elements of a normed vector space X converges to an element $x \in X$ if

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

In such a case, we say that $(x_n)_{n\geq 1}$ is a convergent sequence.

Remark. In a normed linear space, every convergent sequence is a Cauchy sequence.

Definition 1.1.6 *A normed linear space X is complete if every Cauchy sequence of X is convergent in X. A complete normed linear space is called a Banach space.*

Remarks. Every normed linear space has a completion. The notion of completeness is also defined for metric spaces which need not have any linear structure.

Example (Banach spaces). The normed linear space $\big(\mathcal{C}([0,1]);\, \|\cdot\|_\infty\big)$ is a Banach space. Also the space of all bounded linear maps from $\mathbb{R}$ to $\mathbb{R}$ denoted by $\mathcal{B}(\mathbb{R})$ is a Banach space.
The completion of the normed linear space $\big(\mathcal{C}([0,1]);\, \|\cdot\|_2\big)$ where $\|\cdot\|_2$ is defined by

$$\|f\|_2 = \Big(\int_0^1 |f(t)|^2 dt\Big)^{\frac{1}{2}}$$

is $L^2(0,1)$ (see Definition 1.1.7).

1.1.1 Linear operators

In this section X and Y are normed linear spaces over a field $\mathbb{K}$.

Definition 1.1.7 *A $\mathbb{K}$-linear operator T from X into Y is a map $T : X \longrightarrow Y$ satisfying the following property*

$$T(\alpha x + \beta y) = \alpha T x + \beta T y$$

for all α, $\beta \in \mathbb{K}$ and all x, $y \in X$.

When $Y = \mathbb{K}$, such a map is called a linear functional or a linear form.

Proposition 1.1.1 *The set of $\mathbb{K}$-linear maps from X into Y has a natural structure of linear space over $\mathbb{K}$ and is denoted by $\mathcal{L}(X, Y)$. Note that $\mathcal{L}(X, X)$ is simply denoted by $\mathcal{L}(X)$.*

Proposition 1.1.2 *: The sum of two linear maps is a linear map, and the product of a linear operator by a scalar is also a linear operator. Let X, Y and Z be linear spaces. Then*

$$f \in \mathcal{L}(X,Y) \;\; and \;\; g \in \mathcal{L}(Y,Z) \quad \Longrightarrow \quad gof \in \mathcal{L}(X,Z) \;.$$

Theorem 1.1.2 *Let $T \in \mathcal{L}(X,Y)$. Then the following are equivalent*

i) *T is continuous at the origin (in the sense that if $\{x_n\}_n$ is a sequence in X such that $x_n \to 0$ as $n \to \infty$, then $T(x_n) \to 0$ in Y as $n \to \infty$).*

ii) *T is Lipschitz, i.e., there exists a constant $K \geq 0$ such that for every $x \in X$,*

$$||T(x)|| \;\leq\; K||x||\,.$$

iii) *The image of the closed unit ball, $T\big(\overline{B}_1(0)\big)$, is bounded.*

Definition 1.1.8 *A linear operator $T : X \longrightarrow Y$ is said to be bounded if there exist some $k \geq 0$ such that*

$$\|T(x)\| \leq \; k\|x\|$$

for all $x \in X$.

If T is a bounded linear map, then the norm of T is defined by

$$\|T\| \;=\; \inf\big\{k: \;\|T(x)\| \leq \; k\|x\|, \forall x \in X\big\}.$$

The set of bounded linear operators from X into Y is denoted $\mathcal{B}(X,Y)$.

If $X = Y$, one simply writes $\mathcal{B}(X)$ for $\mathcal{B}(X,X)$.

Proposition 1.1.3 *Suppose $X \neq \{0\}$ and $T \in \mathcal{B}(X,Y)$, then we have the following:*

$$\|T\| \;=\; \sup_{\|x\|_X \leq 1} \|T(x)\|_{_Y} \;=\; \sup_{\|x\|_X = 1} \|T(x)\|_{_Y} \;=\; \sup_{x\neq 0} \frac{\|T(x)\|_{_Y}}{\|x\|_{_X}}$$

1.1.2 Differentiability in Banach spaces

Let X and Y be two real Banach spaces and U be a subset of X with nonempty interior.

Definition 1.1.9 (Directional Differentiability) *Let $f : U \longrightarrow Y$ be a map. Let $x_0 \in \mathring{U}$ and $v \in X \setminus \{0\}$.*
The function f is said to be differentiable at x_0 in the direction of the vector v if the function $t \longmapsto f(x_0 + tv)$ is differentiable at $t = 0$. That is, the function

$$t \longmapsto \frac{f(x_0 + tv) - f(x_0)}{t}, \qquad t \neq 0$$

has a limit in the normed linear space Y when t tends to 0 .
This limit, when it exists is denoted by $f'(x_0; v)$ or $\frac{\partial f}{\partial v}(x_0)$.

Note that since $x_0 \in \mathring{U}$, if t is sufficiently close to 0, then $x_0 + tv \in U$ and $f(x_0 + tv)$ is well defined.

Definition 1.1.10 (Gâteaux Differentiability) *Let $f : \quad U \longrightarrow Y$ be a map . Let $x_0 \in \mathring{U}$. The function f is said to be Gâteaux Differentiable at x_0 if :*

1. *f is differentiable at x_0 in every direction $v \in U \setminus \{0\}$ and*

2. *There exists a bounded linear map $A \in \mathcal{B}(X, Y)$ such that $f'(x_0, v) = A(v)$ for all v element of $X \setminus \{0\}$.*

In this case the map $f'(x_0, .)$ is called the Gâteaux differential of f at x_0 and is denoted by $D_G f(x_0)$ or $f'_G(x_0)$.

In orther words, f is Gâteaux differentiable at x_0 if there exists a bounded linear map $A \in \mathcal{B}(X, Y)$ such that

$$\lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0)}{t} = A(v), \qquad \forall v \in X \setminus \{0\}$$

From the definition 1.1.10 it is obvious that if a function is Gâteaux differentiable at a point, then it has a directional derivative in all direction at that point. The following example shows us that the converse is not true.

Example 1 (Gâteaux-differentiable functions)
Let f be the function defined from $\mathbb{R}^2$ into $\mathbb{R}$ by

$$f(x_1, x_2) \;=\; \begin{cases} \frac{x_1^3}{x_1^2 + x_2^2} & \text{if } (x_1, x_2) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}$$

(1) The function f has the directional derivative at the point $0 = (0, 0)$ in all direction.
Indeed, for any $v = (v_1, v_2) \in \mathbb{R}^2 \setminus \{0\}$ we have $\frac{f(tv) - f(0)}{t} = f(v)$.
So $f'(0; v) = \lim_{t \to 0} \frac{f(tv) - f(0)}{t} = f(v)$
(2) The function f is not Gâteaux differntiable at the point 0 since the function $v \mapsto f'(0; v) = f(v)$ is not linear.

Remark: A function $f : U \to Y$ is Gâteaux differentiable at $x_0 \in \mathring{U}$ if there exists a bounded linear mapping $A : X \to Y$ such that $\lim_{t \to 0^+} \frac{f(x_0 + th) - f(x_0)}{t} = A(h), \forall h \in X$.

Definition 1.1.11 (Fréchet Differentiability) *Let $f : \quad U \longrightarrow Y$ be a map. The function f is said to be Fréchet differentiable at $p \in \mathring{U}$ if there exists a bounded linear mapping $T \in \mathcal{B}(X, Y)$ such that*

$$\lim_{h \to 0} \frac{||f(p + h) - f(p) - T(h)||}{||h||} \;=\; 0\,.$$

The bounded linear map $T \in \mathcal{B}(X, Y)$ in this definition is uniquely determined and called the Fréchet *differential of f at p* and it is denoted by $\mathcal{D}f(p)$ or $f'(p)$ (sometine it is also denoted by df_p).

Remark: The function f is said to be Fréchet differentiable on $\mathring{U}$ if it is Fréchet differentiable at every point of $\mathring{U}$. In case the domain of f is open and there is no ambiguity about it we just say that f is differentiable.
It is easy to see that if f is Fréchet differentiable at $x \in \mathring{U}$, then it is continuous at x.

Proposition 1.1.4 *Let $f : U \longrightarrow Y$ be a differentiable map at $p \in \mathring{U}$. Then*

$$df_p(x) \;=\; \lim_{t\to 0} \frac{f(p+tx)-f(p)}{t}, \qquad \forall x \in X.$$

Proof Let $x \in X$. If $x = 0$, obviously proposition 1.1.4 holds. Suppose that $x \neq 0$.
For $t \neq 0$ sufficiently small, we have:

$$\begin{aligned} \left\| df_p(x) - \tfrac{f(p+tx)-f(p)}{t} \right\| &= \left\| \tfrac{df_p(tx)-f(p+tx)+f(p)}{t} \right\| \\ &= ||x|| \cdot \left\| \tfrac{df_p(tx)-f(p+tx)+f(p)}{||tx||} \right\|. \end{aligned}$$

Since f is differentiable at p, we have

$$\lim_{t\to 0} \left\| \frac{df_p(tx)-f(p+tx)+f(p)}{||tx||} \right\| \;=\; 0$$

Therefore we have

$$df_p(x) \;=\; \lim_{t\to 0} \frac{f(p+tx)-f(p)}{t}.$$

Theorem 1.1.3 *If the map $f : U \longrightarrow Y$ is Fréchet differentiable at $p \in \mathring{U}$, then it is Gâteaux differentiable at p and we have $df_p = f'_G(p)$.*

Proof
From the Proposition 1.1.4 we have :

$$df_p(x) = \lim_{t\to 0} \frac{f(p+tx)-f(p)}{t} \qquad \forall x \in X$$

This implies $f'(p;x) = df_p(x), \quad \forall x \in X$
Since df_p is a bounded linear map, then the function f is Gâteaux differentiable at p and $df(p) = f'_G(p)$

The following counterexample, shows that the converse of Theorem 1.1.3 is not true.

Example
Let f be the function defined from $\mathbb{R}^2$ into $\mathbb{R}$ by :

$$f(x_1,x_2) \;=\; \begin{cases} \frac{x_1 x_2^4}{x_1^2+x_2^8} & \text{if } (x_1,x_2) \neq (0,0) \\ 0 & \text{Otherwise} \end{cases}$$

The function f is Gâteaux differentiable at $(0,0)$ but not Fréchet differentiable at $(0,0)$.
Indeed, let $x=(x_1,x_2)\in\mathbb{R}^2\setminus\{0\}$ and $t\neq 0$, we have :

$$\frac{f(tx)-f(0)}{t} = \begin{cases} 0 & \text{if } x_1=0 \\ \frac{t^2x_1x_2^4}{x_1^2+t^6x_2^8} & \text{otherwise} \end{cases}$$

Thus implies that

$$\lim_{t\to 0}\frac{f(tx)-f(0)}{t}=0.$$

Hence f is Gâteaux differentiable at $(0,0)$ and $f'_G(0)=0$.
However on the curve $x_1=x_2^4$, we have $f(x)=\frac{1}{2}$ which does not tend to zero as $x_2\to 0$. So the function f is not continuous at $(0,0)$. Hence it is not Fréchet differentiable at $(0,0)$
This example shows us that even the Gâteaux differentiability does not guarrantee the continuity.

Theorem 1.1.4 (Mean Value Theorem) *Let a, b be two points in a nonempty open set $\Omega\subset X$ such that the segment $[a,b]$ is contained in Ω, and let $f:\Omega\to Y$ be differentiable such that $\sup_{z\in[a,b]}||f'(z)||<\infty$. Then*

$$||f(a)-f(b)||\leq M||a-b||,$$

where $M=\sup_{z\in[a,b]}||f'(z)||$.

Proof
Let $a,b\in\Omega$ such that $[a,b]\subset\Omega$ and $g\in Y^*$, with $||g||\leq 1$. Consider the map $\phi:[0,1]\to X$ defined by :

$$\phi(t)=g\circ f(a+t(b-a)) \qquad \forall t\in[0,1]$$

Since f is differentiable, ϕ is also differentiable and we have :

$$\phi'(t) \;=\; g(f'(a+t(b-a))(b-a))$$

with

$$\sup_{s\in(0,1)}||\phi'(s)|| \;\leq\; ||g||\sup_{z\in[a,b]}||f'(z)||\,||b-a|| \;\leq\; M||b-a|| \;<\; \infty\,.$$

Therefore by the classical Mean Value Theorem in $\mathbb{R}$, we have

$$\begin{aligned} ||\phi(1)-\phi(0)|| &\leq \sup_{0\leq t\leq 1}|\phi'(t)| \\ &\leq M||b-a||, \end{aligned}$$

which yields the desired result by the Hahn-Banach Theorem since for every $x\in X$,

$$||x|| \;=\; \sup\big\{||T(y)||\;:\;T\in Y^* \text{ and } ||T||\leq 1\big\}.$$

.

Definition 1.1.12 *Let $U\subset X$ be a nonempty open set and $f:U\longrightarrow Y$ be Fréchet differentiable mapping. The differential of f on U is the mapping*

$$\begin{array}{cccc} Df: & U & \longrightarrow & \mathcal{B}(X,Y) \\ & x & \longmapsto & Df(x) \end{array}$$

We say that f is continously differentiable on U (or of class C^1) if Df is a continuous mapping.

The following theorem gives a sufficient condition for a Gâteaux differentiable function to be Fréchet differentiable.

Theorem 1.1.5 *Assume that $f : U \longrightarrow F$ is Gâteaux differentiable on U and such that f'_G is continuous from U into $\mathcal{B}(X,Y)$. Then f is Fréchet differentiable and its Fréchet and Gâteaux differentials are the same.*

Proof
Let $p \in U$. Then there exists $\epsilon > 0$ such that $B(p, \epsilon) \subset U$. Let $x \in X$ such that $||x|| < \epsilon$. Consider the map $\phi : [0,1] \longrightarrow Y$ defined by :

$$\phi(t) = f(p + tx) - f(x) - tf'_G(p)(x)$$

Since f is Gâteaux differentiable, it follows that ϕ is differentiable and

$$\phi'(t) = f'_G(p + tx)(x) - f'_G(p)(x).$$

Observe that ϕ' is bounded and by applying the mean value theorem to ϕ we have:

$$|\phi(0) - \phi(1)| \leq \sup_{0 \leq t \leq 1} |\phi'(t)|$$

This is equivalent to

$$\begin{array}{rcl} ||f(p + x) - f(p) - f'_G(p)(x)|| & \leq & \sup_{0\leq t\leq 1} ||f'_G(p + tx)(x) - f'_G(p)(x)|| \\ & \leq & \sup_{0\leq t\leq 1} ||f'_G(p + tx) - f'_G(p)||||x|| \end{array}$$

Let $\lambda > 0$. Then by continuity of the mapping $u \mapsto f'_G(u)$, there exists $\gamma > 0$ such that

$$||p - y|| < \gamma \Rightarrow ||f'_G(y) - f'_G(p)|| < \lambda.$$

If $||x|| < \min\{\epsilon, \gamma\}$, then for every $t \in [0,1]$, $p + tx \in B(p, \gamma)$. This implies that

$$||f'_G(p + tx) - f'_G(p)|| < \lambda \quad \forall t \in [0,1]\,.$$

Hence $\sup\limits_{0\leq t\leq 1} ||f'_G(p + tx) - f'_G(p)| \leq \lambda$. Since λ is arbitrarily choosen, we deduce that:

$$\lim_{x\to 0} \big(\sup_{0\leq t\leq 1} ||f'_G(p + tx) - f'_G(p)||\big) = 0$$

implying that

$$\lim_{x\to 0} \frac{||f(p + x) - f(p) - f'_G(p)(x)||}{||x||} = 0$$

Hence the function f is Fréchet differentiable at p. We are done.

1.1.3 Fundamental theorems

In this section we state some fundamental theorems that will be useful in the study of our theory.

Definition 1.1.13 *Let X and Y be Banach spaces.*

i) *A function $g : [0,T] \times \mathbf{X} \to \mathbf{Y}$ is globally Lipschitz on $D \subseteq \mathbf{X}$ (uniformly in t), if there exists a positive constant $K = K(g)$ (independent of t) such that*

$$\|g(t,x) - g(t,y)\|_{\mathbf{Y}} \;\leq\; K\|x-y\|_X, \quad \forall t \in [0,T] \quad \text{and} \quad x, y \in \mathbf{X}.$$

A function $f : \mathbf{X} \to \mathbf{Y}$ is locally Lipschitz on $\mathbf{X}$, if $\forall x_0 \in \mathbf{X}$ and $\epsilon > 0$, there exists $k = k(x_0, \epsilon)$ such that

$$\|f(x) - f(y)\| \;\leq\; k\|x-y\|, \quad \forall x, y \in \mathcal{B}_{\mathbf{X}}(x_0, \epsilon).$$

ii) *A mapping $\phi : \mathbf{X} \to \mathbf{X}$ is a contraction, if there exists $0 < \alpha < 1$ such that*

$$\|\phi(x) - \phi(y)\| \;\leq\; \alpha\|x-y\|, \quad \forall x, y \in \mathbf{X}.$$

Theorem 1.1.6 (Contraction Mapping theorem) *If $\phi : \mathbf{X} \to \mathbf{X}$ is a contraction, then there exists a unique $z \in \mathbf{X}$ such that $\phi(z) = z$ (i.e., ϕ has a unique fixed-point).*

Theorem 1.1.7 (Uniform Boundedness Principle) *Let X and Y be Banach spaces. Suppose that*

i) $\{T_\alpha\}_{\alpha\in\Delta} \subset \mathcal{B}(X,Y)$;

ii) $\forall x \in X, \;\; \sup_{\alpha\in\Delta} \|T_\alpha(x)\| < \infty$.

Then $\sup\limits_{\alpha\in\Delta} \|T_\alpha\| \;<\; \infty.$

Theorem 1.1.8 (Closed-graph Theorem) *Let X and Y be two Banach spaces. Let T be a linear map from X into Y. Assume that the graph of T, G_T, is closed in $X \times Y$. Then T is continuous.*

Definition 1.1.14 **i)** *Let $\mathbf{X}$ be a Banach space and D a subset of $\mathbf{X}$. We say that D is precompact if the closure of D in $\mathbf{X}$ is compact.*

ii) *A set $\mathcal{F} \subset \mathcal{C}([a,b], \mathbf{X})$ is equicontinuous at $t_0 \in [a,b]$, if $\forall \epsilon > 0, \exists \delta > 0$ (depending on t_0 and ϵ) such that:*
$t \in [a,b]$ *with* $\mid t - t_0 \mid < \delta \Rightarrow \|f(t_0) - f(t)\| < \epsilon, \quad \forall f \in \mathcal{F}$.

iii) *Let $\mathbf{X}$ and $\mathbf{Y}$ be Banach spaces. An operator $A : \mathbf{X} \to \mathbf{Y}$ is compact if for every bounded set $D \subset \mathbf{X}$, the image $A(D)$ is precompact in $\mathbf{Y}$.*

Definition 1.1.15 *A sequence of continuous functions* $f_n : [a,b] \to \mathbb{R}$ *(i.e.,*$f_n \in \mathcal{C}([a,b])$*) is called equicontinuous if,*

$$\forall\, \epsilon > 0, \quad \exists \delta > 0 : \quad |s-t| < \delta \ \Rightarrow \ |f_n(s) - f_n(t)| \ < \ \epsilon \ \text{ for any } n \geq 1.$$

Theorem 1.1.9 (Arzela-Ascoli Theorem.) *Any bounded equicontinuous sequence* $\{f_n\}$ *in* $\mathcal{C}([a,b])$ *has a uniformly convergent subsequence.*

We now state a version of the above theorem in $\mathcal{C}([a,b], X)$ which will be useful in the dissertation.

Theorem 1.1.10 ***[Arzela-Ascoli in*** $\mathcal{C}([a,b],\mathbf{X})$***]*** *A set* $\mathcal{K} \subset \mathcal{C}([a,b],\mathbf{X})$ *is precompact if and only if* $\forall t \in [a,b]$ *the set* $\{z(t) | z \in \mathcal{K}\} \subset \mathbf{Y}$ *is precompact in* Y *and equicontinuous at* t*.*

The following theorem will be used to prove the existence of a mild solution of the Cauchy problem, when the forcing term does not satisfy a Lipschitz-type condition, and A generates a compact semigroup.

Theorem 1.1.11 (Schaefer's Fixed-point Theorem) . *Let* X *be a Banach space and let* $\psi : \mathbf{X} \to \mathbf{X}$ *be a continuous, compact operator and let*

$$\xi(\psi) \ = \ \big\{ x \in \mathbf{X} \ : \ \lambda x = \psi(x) \ for\ some\ \lambda \geq 1 \big\}.$$

If $\xi(\psi)$ *is bounded, then* ψ *has at least one fixed-point.*

Expenential Maps

We define and give some regularity properties of the exponential map with respect to bounded linear operators.

Let E be Banach space throughout this section. Then it is not hard to see that for every bounded linear operator A on E, the series

$$I + \sum_{n=1}^{\infty} \frac{A^n}{n!}$$

converges absolutely in $\mathcal{B}(E)$. For simplicity we write

$$\sum_{n=0}^{\infty} \frac{A^n}{n!}$$

for

$$I + \sum_{n=1}^{\infty} \frac{A^n}{n!}$$

and we have the following

Definition 1.1.16 *The exponential of any bounded linear operator* A *is defined by*

$$exp(A) = e^A \ := \ \sum_{n=0}^{\infty} \frac{A^n}{n!} \,.$$

Theorem 1.1.12 *We have the following*

i) $e^0 = I$.

ii) $A^m e^A = e^A A^m$, *for all* $A \in \mathcal{B}(E)$ *and all natural number* m.

iii) *For all* $A, B \in \mathcal{B}(E)$ *such that* $AB = BA$,
we have $Ae^B = e^B A$ *and* $e^A e^B = e^B e^A = e^{A+B}$.

Corollary 1.1.1 *Let* $A \in \mathcal{B}(E)$ *and let* $t, s \in \mathbb{R}$. *Then*

$$e^{(t+s)A} = e^{sA} e^{tA}$$

Proof of the Theorem 1.1.12: We prove only (iii) since it implies (ii) while (i) follows immediately from the definition. Assume that $A, B \in \mathcal{B}(E)$ and satisfy $AB = BA$. Therefore

$$\begin{aligned}
Ae^B &= A \sum_{k=0}^{\infty} \frac{B^k}{k!} \\
&= A \left(\lim_{n\to\infty} \sum_{k=0}^{n} \frac{B^k}{k!} \right) \\
&= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{AB^k}{k!}, \quad \text{by continuity and linearity of } A \\
&= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{B^k A}{k!}, \quad \text{by commutativity of } A \text{ and } B \\
&= \left(\sum_{k=0}^{\infty} \frac{B^k}{k!} \right) A \\
&= e^B A
\end{aligned}$$

Moreover it follows, by mathematical induction, that for all $k \in \mathbb{N}$ we have $A^k e^B = e^B A^k$. Thus

$$\begin{aligned}
e^A e^B &= \left(\sum_{k=0}^{\infty} \frac{A^k}{k!}\right) e^B \\
&= \lim_{m\to\infty} \left(\sum_{k=0}^{m} \frac{A^k}{k!}\right) e^B \\
&= \lim_{m\to\infty} \sum_{k=0}^{m} \frac{A^k}{k!} e^B \\
&= \lim_{m\to\infty} \sum_{k=0}^{m} \frac{A^k}{k!} \sum_{p=0}^{\infty} \frac{B^p}{p!} \\
&= \lim_{m\to\infty} \sum_{k=0}^{m} \sum_{p=0}^{\infty} \frac{A^k B^p}{k!p!} \quad \text{by continuity and linearity of } A \\
&= \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \frac{A^k B^p}{k!p!} \\
&= \sum_{n=0}^{\infty} \sum_{k+p=n} \frac{A^k B^p}{k!p!}, \quad \text{since the series converge absolutely} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{A^k B^{n-k}}{k!(n-k)!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{n!} \binom{n}{k} A^k B^{n-k} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n}) \binom{n}{k} A^k B^{n-k} \\
&= \sum_{n=0}^{\infty} \frac{(A+B)^n}{n!} \quad \text{by the commutativity of } A \text{ and } B \\
&= e^{(A+B)}
\end{aligned}$$

Theorem 1.1.13 *Let A be in $\mathcal{B}(E)$, t be a real scalar variable, $x \in \mathbf{E}$ and let*

$$f(t) = e^{tA}x.$$

Then f is differentiable for all $x \in E$ and $f'(t) = Ae^{tA}x$.

Theorem 1.1.14 *The exponential map is continuously differentiable from $\mathcal{B}(E)$ into $\mathcal{B}(E)$.*

Theorem 1.1.15 (Existence and Uniqueness) *Given any bounded linear operator A on $\mathbf{E}$, for every $U_0 \in \mathbf{E}$, the homogeneous Cauchy problem*

$$\begin{cases} U'(t) &= AU(t), \quad t > 0 \\ U(0) &= U_0 \end{cases} \tag{1.1}$$

has a unique solution U in $\mathcal{C}([0,\infty), \mathbf{E}) \cap \mathcal{C}^1((0,\infty); \mathbf{E})$ expressed by $U(t) = e^{tA}U_0$.

Proof: Let $U_0 \in E$

Existence:

We must verify that

$$U(t) = e^{tA}U_0$$

is continuous on $[0, T]$, differentiable on $(0, T]$, and satifies (1.1).
But we know that for every $x_0 \in E$, $g : [0, \infty) \longrightarrow E$ defined by

$$g(t) = e^{tA}x_0$$

is continuous. So $U \in \mathcal{C}([0, \infty); E)$ and since $g \in \mathcal{C}^1([0, \infty); E)$ this ensures that $U \in \mathcal{C}^1([0, \infty); E)$ and U satisfies the O.D.E in (1.1) and

$$U(0) = e^0 U_0 = U_0$$

satisfying the initial condition. This thus establishes the existence.

Uniqueness

Let $V : [0, \infty) \longrightarrow E$ be a solution to the Initial value Problem:

$$\begin{cases} V'(t) &= AV(t), \quad t > 0 \\ V(0) &= V_0 \end{cases}$$

Define the function $\psi : [0, \infty) \longrightarrow E$ by

$$\psi(s) = e^{(t-s)A}V(s)$$

. Since $t \mapsto e^{tA}$ is differentiable $\forall t \geq 0$ so is ψ. So let $t_0 > 0$. Then $\forall s \in [0, t_0]$

$$\begin{aligned} \frac{d\psi}{ds}(s) &= e^{(t_0-s)A}V'(s) + \frac{d}{ds}e^{(t_0-s)A}V(s) \\ &= e^{(t_0-s)A}V'(s) - e^{(t_0-s)A}AV(s) \\ &= e^{(t_0-s)A}V'(s) - e^{(t_0-s)A}V'(s) \\ &= 0 \end{aligned}$$

So ψ is a constant function and therefore $\psi(0) = \psi(t_0), \forall t_0 > 0$ i.e, $U(t_0) = V(t_0)$ showing uniqueness and therefore completing the proof.

Remark: The above results clearly hold when E is $\mathbb{R}^N$ (finite dimensional space) and A is a Square matrix of order N.

1.1.4 Riemann Integration of functions with values in Banach spaces

Here we shall define the Riemann integral for abstract functions of the real variable and discuss some properties which will be constantly used in this project. By an abstract function we mean a function x defined on an interval $[a, b]$ of $\mathbb{R}$ $(a \leq b)$ with values in a Banach space X. Let $x : [a, b] \to X$ be an abstract function.

We say that x is continuous at $t_0 \in [a, b]$ if

$$\|x(t) - x(t_0)\| \longrightarrow 0 \quad \text{as} \quad t \to t_0 \quad \text{in } [a, b].$$

The function x is continuous on $[a,b]$ if it is continuous at each point of $[a,b]$. We denote the partition

$$a = t_0 < t_1 < t_2 < ... < t_n = b$$

together with chosen points

$$\tau_i \in [t_i,\, t_{i+1}], \quad i = 0, 1, 2, \, ..., \, n-1$$

by π and set $|\pi| = \max_i |t_{i+1} - t_i|$.
We form the Riemann sum associated to π

$$S_\pi = \textstyle\sum_{i=0}^{n-1} (t_{i+1} - t_i) x(\tau_i)$$

By varying π, if S_π has a limit I in X, as $|\pi| \to 0$, then I is called the *Riemann integral* of the function x and is denoted by

$$I \ = \ \int_a^b x(t)\, dt\, .$$

In such a case we also define

$$\int_b^a x(t)\, dt \ := \ -\int_a^b x(t)\, dt\, .$$

Theorem 1.1.16 *Let $a < b$ in $\mathbb{R}$. If $x \in \mathcal{C}([a,b], X)$, then the Riemann integral $\int_a^b x(t)dt$ exists.*

Properties
Using the definition of the Riemann integral one can easily verify the following properties for a continuous function $x : \, [a,b] \to X$ where $a < b$ are real numbers.

i) $\int_a^b x(t)dt \ = \ \int_a^c x(t)dt + \int_c^b x(t)dt$, for $a < c < b$ provided that one of the integrals on the right exists.

ii) If $x(t) = x_0$ for all $t \in [a,b]$, we have $\int_a^b x_0 dt = (b-a)x_0$

iii) If $t = w(\tau)$ is an increasing continuously differentiable function on $[\alpha, \beta]$ with $a = w(\alpha)$ and $b = w(\beta)$, then

$$\int_a^b x(t)dt \ = \ \int_\alpha^\beta x[w(\tau)]w'(\tau)d\tau$$

v) $\| \int_a^b x(t)dt \| \leq \int_a^b \|x(t)\| dt$.

Indeed, from the definition of the Riemann sum we have

$$\begin{aligned} \|S_\pi\| \ &\leq \ \Big\| \sum_{i=0}^{n-1} (t_{i+1} - t_i) x(\tau_i) \Big\| \\ &\leq \ \sum_{i=0}^{n-1} (t_{i+1} - t_i) \|x(\tau_i)\| \longrightarrow \int_a^b \|x(t)\| dt, as \quad |\pi| \longrightarrow 0 \end{aligned}$$

since $\|x(t)\|$ is continuous and hence is integrable on $[a,b]$.

Theorem 1.1.17 (Dominated convergence theorem) *If $\{x_n\}_{n\geq 1}$ is a sequence of abstract measurable functions which converges to an abstract function x on the interval $[a,b]$, and $\|x_n(t)\| \leq y(t)$ almost everywhere on $[a,b]$ for all n, where y is a nonnegative integrable function on $[a,b]$. Then*

$$\lim_{n\longrightarrow\infty}\int_a^b x_n(t)dt \;=\; \int_a^b x(t)dt.$$

Proof: We have

$$\begin{aligned}\Big\|\int_a^b x_n(t)dt - \int_a^b x(t)dt\Big\| &\leq \int_a^b \|x_n(t)-x(t)\|dt\\ &\leq \max_{[a,b]}\|x_n(t)-x(t)\|(b-a) \longrightarrow 0 \quad \text{as} \quad n\longrightarrow\infty\end{aligned}$$

which proves the desired result.

Theorem 1.1.18 *If $x \in \mathcal{C}([a,b],X)$, then*

$$\frac{d}{dt}\left(\int_a^t x(s)ds\right) \;=\; x(t), \quad a\leq t\leq b.$$

Proof: Set

$$y(t) = \int_a^t x(s)ds.$$

Then in view of the fact that $x(t)$ is uniformly continuous on $[a,b]$ as a continuous function on a closed and bounded interval, we have

$$\begin{aligned}\Big\|\frac{y(t+h)-y(t)}{h} - x(t)\Big\| &= \Big\|\frac{1}{h}\Big[\int_a^{t+h} x(s)ds - \int_a^t x(s)ds\Big] - x(t)\Big\|\\ &= \Big\|\frac{1}{h}\Big[\int_a^{t+h} x(s)ds + \int_t^a x(s)ds - \int_t^{t+h} x(t)ds\Big]\Big\|\\ &\quad \text{by properties (i) and (iii)}\\ &= \Big\|\frac{1}{h}\int_t^{t+h}\big[x(s)-x(t)\big]ds\Big\|, \quad \textit{by} \quad \text{property} \quad \text{(ii)}\\ &\leq \max_{|s-t|\leq|h|}\|x(s)-x(t)\| \longrightarrow 0 \quad \text{as} \quad h\longrightarrow 0\end{aligned}$$

and the proof is complete.

Theorem 1.1.19 (Fundamental theorem of calculus in Banach spaces) *If the function $x:[a,b]\to X$ is continously differentiable on (a,b), then for any $\alpha,\beta\in(a,b)$ the following formula is true:*

$$\int_\alpha^\beta x'(s)ds \;=\; x(\beta) \;-\; x(\alpha)$$

Proof: By theorem 1.1.18
$(\frac{d}{dt})\left[\int_\alpha^t x'(s)ds \; - \; x(t)\right] \; = \; 0, \alpha \leq t \leq \beta$. Hence

$$\int_\alpha^t x'(s)ds \; - \; x(t) \; = \; \text{const}. \tag{1.2}$$

For $t = \alpha$ we find the value $const = -x(\alpha)$ and the result follows by setting $t = \beta$ in the above identity.

The following theorem which asserts that integration commutes with closed operators (in particular, integration commutes with bounded linear operators) will be useful later in the project.

Theorem 1.1.20 *Let A on $\mathcal{D}(A)$ be a closed operator in the Banach space X and $x \in \mathcal{C}([a,b), X)$ with $b \leq \infty$. Suppose that $x(t) \in \mathcal{D}(A)$, $Ax(t)$ is continuous on $[a,b)$ and that the improper integral $\int_a^b x(t)dt$ and $\int_a^b Ax(t)dt$ exist. Then*

$$\int_a^b x(t)dt \in \mathcal{D}(A)$$

and

$$A\int_a^b x(t)dt = \int_a^b Ax(t)dt$$

1.1.5 Gronwall Lemma, Differential Inequality

We now state a result (where the above Riemann integral is used) of which we will make use in the sequel.

Lemma 1.1.1 (Gronwall's lemma) *Let $t_0 \in (-\infty, T)$, $x \in \mathcal{C}([t_0,T], \mathbb{R})$, $K \in \mathcal{C}([t_0,T], [0,+\infty))$ and M be a real constant.*
Suppose $x(t) \leq M + \int_{t_0}^t K(s)x(s)ds, \quad t_0 \leq t \leq T$.
Then

$$x(t) \; \leq \; Me^{\int_{t_0}^t K(s)ds}, \quad t_0 \leq t \leq T.$$

Proof Define $y : [t_0, T] \to \mathbb{R}$ by

$$y(t) = M + \int_{t_0}^t K(s)x(s)ds.$$

Observe that $y \in \mathcal{C}^1([t_0,T], \mathbb{R})$ and satisfies the IVP

$$\left\{\begin{array}{lcl} y'(t) & = & K(t)x(t), \quad t_0 \leq t \leq T \\ y(t_0) & = & M \end{array}\right. \tag{1.3}$$

By assumption, $x(t) \leq y(t), \quad t_0 \leq t \leq T$. Multiplying both sides of this inequality by K(t) and then substituting into the above IVP yields

$$\left\{\begin{array}{lcl} y'(t) & \leq & K(t)y(t), \quad t_0 \leq t \leq T \\ y(t_0) & = & M \end{array}\right. \tag{1.4}$$

Hence

$$y'(t) - K(t)y(t) \leq 0, \quad t_0 \leq t \leq T,$$

so that multiplying both members by $e^{-\int_{t_0}^{t} K(s)ds}$ yields

$$\frac{d}{dt}\left[e^{-\int_{t_0}^{t} K(s)ds} y(t)\right] \; \leq \; 0, \quad t_0 \leq t \leq T$$

Consequently,

$$e^{-\int_{t_0}^{t} K(s)ds} y(t) \leq y(t_0) = M, \quad t_0 \leq t \leq T,$$

and so $y(t) \; \leq \; Me^{\int_{t_0}^{t} K(s)ds}$, $\quad t_0 \leq t \leq T$. Since $x(t) \leq y(t)$, $\quad t_0 \leq t \leq T$, the conclusion follows from the above inequality.

1.1.6 Function Spaces with Values in a Banach Space

Let X be a Banach space with norm $\|.\|$. We introduce various function spaces consisting of functions defined on an interval of $\mathbb{R}$ or on a domain of $\mathbb{C}$ with values in X.

Uniformly Bounded Function Spaces

Let [a, b] be a bounded closed interval. By $\mathcal{B}([a,b];X)$ we denote the space of uniformly bounded functions on [a, b] (not necessarily smooth or measurable). The space is a Banach space with the supremum norm

$$\|F\|_{\mathcal{B}} = \sup_{a \leq t \leq b} \|F(t)\|$$

Continuously Differentiable Function Spaces

Let $[a,b]$ be a bounded closed interval, and let $m = 0, 1, 2, ...,$ be a nonnegative integer. $\mathcal{C}^m([a,b];X)$ denotes the space of m times continuously differentiable functions on $[a,b]$. When $m = 0$, $\mathcal{C}^0([a,b];X)$ is the space of continuous functions and is simply denoted by $\mathcal{C}([a,b];X)$. The space is equipped with the maximum norm

$$\|F\|_{\mathcal{C}^m} \;=\; \sum_{i=0}^{m} \max_{a \leq t \leq b} \|F^{(i)}(t)\|\,.$$

Let us recall that for any continuous function $F \in \mathcal{C}([a,b];X)$, the Riemann integral of F on the interval $[a,b]$ is defined and is denoted by $\int_a^b F(t)dt$ and satisfies

$$\left\|\int_a^b F(t)dt\right\| \leq \int_a^b \|F(t)\|dt\,.$$

Hölder Continuous Function Spaces

For $m = 0, 1, 2, \ldots$, and an exponent $0 < \sigma < 1$, $\;\mathcal{C}^{m+\sigma}([a,b];X)$ denotes the space of m-times continuously differentiable functions whose m^{th} derivatives are Hölder continuous on $[a,b]$ with exponent σ. The space is equipped with the norm

$$\|F\|_{\mathcal{C}^{m+\sigma}} \;=\; \|F\|_{\mathcal{C}^m} + \sup_{a \leq s < t \leq b} \frac{\|F^m(t) - F^m(s)\|}{\mid t - s \mid^{\sigma}}$$

L^P and Sobolev Spaces

In the sequel, Ω is a nonempty open subset of $\mathbb{R}^n$ with Lebesgue measure dx.

Definition 1.1.17 *Let $1 \leq p < \infty$. We define:*
(i) $L^p(\Omega)$ as the set of measurable functions $f : \Omega \to \mathbb{R}$ such that:

$$\int_\Omega |f(x)|^p \, dx \; < +\infty$$

and
(ii) $L^\infty(\Omega)$ as the set of measurable functions $f : \Omega \to \mathbb{R}$ such that:

$$ess\ sup\ |f| \; < \; +\infty$$

where

$$ess\ sup\ |f| = inf\{K \geq \ 0, \ \ |f(x)| \leq \ K, \ \ for \ \ a.e\ x \in \ \Omega\}$$

For $f \in L^p(\Omega)$, we define:

$$\|f\|_p \;\; = \;\; \left[\int_\Omega |f(x)|^p \, dx\right]^{\frac{1}{p}}, \;\; 1 \leq p < +\infty \tag{1.5}$$

$$and \; \|f\|_\infty \;\; = \;\; \text{ess sup } |f| \tag{1.6}$$

Theorem 1.1.21 (Holder's Inequality.) *Let $1 \leq p < +\infty$, we define q by $1/p + 1/q = 1$. If $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then $fg \in L^1(\Omega)$ and*

$$\|fg\|_1 \; \leq \|f\|_p \|g\|_q. \tag{1.7}$$

Definition 1.1.18 (Weak Derivative.) *Let $u \in L^1_{\text{loc}}(\Omega)$. For a given multi-index $\alpha \in \mathbb{N}^n$, a function $v \in L^1_{\text{loc}}(\Omega)$ is called the α^{th}-weak derivative of u if*

$$\int_\Omega u \cdot D^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_\Omega v \cdot \varphi \, dx, \;\; \forall \; \varphi \in \; D(\Omega). \tag{1.8}$$

where $D(\Omega)$ is the space of all $\mathcal{C}^\infty$ functions with compact support in Ω, and $L^1_{\text{loc}}(\Omega)$ is the set of all functions that are integrable on any compact subset of Ω.

The function v is also refered to as the *generalized derivative* of u and we write $v = D^\alpha u$. Clearly $D^\alpha u$ is uniquely determined up to set of *Lebesgue* measure zero.

Definition 1.1.19 *Let m be a non-negative integer and $1 \leq p \leq \infty$. The* **Sobolev space** *$W^{m,p}(\Omega)$ is defined by*

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) \;\mid\; D^\alpha u \; \in \; L^p(\Omega), \;\; for\ all \;\; \alpha \; \in \; \mathbb{N}^n \; : \; |\alpha| \; \leq m\}. \tag{1.9}$$

In other words, $W^{m,p}(\Omega)$ is the collection of all functions in $L^p(\Omega)$ for which all weak derivatives up to order m are also in $L^p(\Omega)$. Clearly $W^{m,p}(\Omega)$ is a vector space. (In all that follows we will consider functions with values in $\mathbb{R}$ and the corresponding function spaces as vector spaces over $\mathbb{R}$). We provide it with the norm:

$$\|u\|_{W^{m,p}(\Omega)} = \sum_{|\alpha|\leq m} \|D^\alpha u\|_{L^p(\Omega)}. \tag{1.10}$$

or equivalently, for $1 \leq p < \infty$,

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha|\leq m} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}. \tag{1.11}$$

The cases $p = 2$ will play a special role in this project. These spaces will be denoted by $H^m(\Omega)$. Thus

$$H^m(\Omega) = W^{m,2}(\Omega) \tag{1.12}$$

and for $u \in H^m(\Omega)$, its norm is denoted by

$$\|u\|_{H^m(\Omega)} = \|u\|_{W^{m,2}(\Omega)}. \tag{1.13}$$

The spaces $H^m(\Omega)$ are Hilbert spaces with the natural inner-product defined by

$$(u, v)_{H^m} = \sum_{|\alpha|\leq m} \int_\Omega D^\alpha u D^\alpha v \, dx, \quad \forall\, u,\ v\ \in\ H^m(\Omega). \tag{1.14}$$

This inner-product yields the norm given by formula (1.11).

Finally, we introduce an important subspace of the space $W^{m,p}(\Omega)$. If $1 \leq p < \infty$, we know that $D(\Omega)$ is dense in $L^p(\Omega)$. Also, if $\varphi \in D(\Omega)$, so does every derivative of φ and so $D(\Omega) \subset W^{m,p}(\Omega)$, for any m and p.

If $1 \leq p < \infty$, we define the space $W_0^{m,p}(\Omega)$ as the closure of $D(\Omega)$ in $W^{m,p}(\Omega)$. Thus $W_0^{m,p}(\Omega)$ is a closed subspace of $W^{m,p}(\Omega)$ and its elements can be approximated in the $W^{m,p}(\Omega)$-norm by C^∞ functions with compact support. In general this is a strict subspace of $W^{m,p}(\Omega)$, except when $\Omega = \mathbb{R}^n$.

Sobolev Embedding

The following theorem will be very useful in the study of some particular types of Evolution Equations.

Theorem 1.1.22 (Sobolev embedding theorem) *Let Ω be a bounded open subset of $\mathbb{R}^N$ with $\mathcal{C}^1$ boundary. Then we have*
1) If $1 \leq p < N$ and if $p \leq q \leq p^*$ (where $p^* = \frac{Np}{N-p}$) then $W^{1,p}(\Omega) \subset L^q(\Omega)$ with continuous embedding.

2) *If $p = N$ then $W^{1,p}(\Omega) \subset L^q(\Omega)$ for all $q \in [p, \infty)$ with continuous embedding.*

3) *If $p > N$ then $W^{1,p}(\Omega) \subset L^q(\bar{\Omega})$ with continuous embedding. Moreover for all $u \in W^{1,p}(\Omega)$ and for all $x, y \in \Omega$ we have*

$$|u(x) - u(y)| \ \leq \ C\|u\|_{W^{1,p}(\Omega)}|x - y|^{\theta}, \quad \theta = 1 - \frac{N}{p}.$$

Compact Embedding:

Theorem 1.1.23 (Rellich-Kondrachov) *Let Ω be an open bounded subset of $\mathbb{R}^n$ with $\mathcal{C}^1$ boundary. If $1 \leq p < +\infty$, then $W^{1,p}(\Omega)$ is compactly embedded into $L^q(\Omega)$ if $1 \leq q < p^*$ (where $p^* = \frac{np}{(n-p)}$). That is*
i) There is a constant C depending only on p, n, and Ω such that $\|u\|_{L^q(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$, for every $u \in W^{1,p}(\Omega)$.
ii) Every bounded sequence (u_k) in $W^{1,p}(\Omega)$ has a subsequence (u_{k_j}) that converges in $L^q(\Omega)$.

1.2 Semigroups

In this section we will review some basic concepts and results concerning spectral theory and semigroup of linear operators that will be useful in the dissertation. The material is standard and can be found in any textbook on the subject. So we state the most important results some without proof as any interested reader can find details in [5].

1.2.1 Spectral Theory of linear Operators

Definition 1.2.1 *Let X be a Banach space and $A : \mathcal{D}(A) \subset X \to X$ be a linear operator. Then the resolvent set of A denoted by $\rho(A)$, is defined by*

$$\rho(A) \;=\; \{\lambda \in \mathbb{C} : \; \lambda I - A : \mathcal{D}(A) \to X \quad \textit{has} \quad \textit{a} \quad \textit{bounded} \quad \textit{inverse}\} \,.$$

The spectrum of A denoted $\sigma(A)$ is the complement of $\rho(A)$ in $\mathbb{C}$.
For $\lambda \in \rho(A)$, the inverse

$$R(\lambda, A) := (\lambda I - A)^{-1} \tag{1.15}$$

is (by definition) a bounded linear operator on X and will be called the resolvent of A at λ.

Remark 1.2.1 **i)** *It follows from the definition above that the identity*

$$AR(\lambda, A) = \lambda R(\lambda, A) - I \tag{1.16}$$

holds for every $\lambda \in \rho(A)$ and $R(\lambda, A)$ commutes with A.

ii) *For λ, $\mu \in \rho(A)$ one has:*

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\mu, A)R(\lambda, A) \tag{1.17}$$

and

$$R(\lambda, A)R(\mu, A) = R(\mu, A)R(\lambda, A) \tag{1.18}$$

iii) *If $\rho(A) \neq \emptyset$, then A is closed.*

Theorem 1.2.1 *Let $A : \mathcal{D}(A) \subset X \to X$ be a closed linear operator. Then the following properties hold.*

i) *The resolvent set $\rho(A)$ is open in $\mathbb{C}$, and for $\mu \in \rho(A)$ one has*

$$R(\lambda, A) = \sum_{n=0}^{\infty} (\mu - \lambda)^n R(\mu, A)^{n+1} \tag{1.19}$$

for $\lambda \in \mathbb{C}$ such that $|\mu - \lambda| < \frac{1}{\|R(\mu,A)\|}$

ii) *For every $\lambda \in \rho(A)$, one has*

$$\|R(\lambda, A)\| \geq \frac{1}{dist(\lambda, \sigma(A))} \tag{1.20}$$

iii) *Let $(\lambda_n)_{n\in\mathbb{N}} \subset \rho(A)$ with $\lambda_n \to \lambda_0$ as $n \to \infty$.*
Then $\lambda_0 \in \sigma(A)$ if and only if $\lim_{n\to\infty} \|R(\lambda_n, A)\| = \infty$

Note that if A is bounded, then since

$$R(\lambda, A) = \frac{1}{\lambda}(1 - \frac{A}{\lambda})^{-1} \;=\; \sum_{n=0}^{\infty} \frac{A^n}{\lambda^{n+1}}, \quad \text{for all } \lambda \text{ such that } |\lambda| > \|A\|. \tag{1.21}$$

It follows that

$$\sigma(A) \;\subset\; \{\lambda \in \mathbb{C} : |\lambda| \;\leq\; \|A\|\}\,.$$

Corollary 1.2.1 *For a bounded linear operator A on X, the spectrum $\sigma(A)$ is always compact and nonempty; hence its spectral radius defined by*

$$r(A) \;=\; \sup\{|\lambda| :\; \lambda \in \sigma(A)\}$$

is finite and satisfies

$$r(A) \;\leq\; \|A\|\,.$$

Theorem 1.2.2 (Gelfand Formula) *If*

$$A \in \mathcal{B}(X) \quad \text{then} \quad r(A) \;=\; \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}.$$

When A is a closed operator, we have, by the Closed Graph Theorem, the following equivalent definition of the resolvent

$$\rho(A) \;=\; \{\lambda \in \mathbb{C} :\; \lambda I - A : \mathcal{D}(A) \to X \quad \textit{is bijective} \quad \}$$

and so from the definition of the spectrum of the operator A now characterized by the non-bijectivity of $\lambda I - A$, we can distinguish three subsets of the spectrum set $\sigma(A)$ summarized in the following definition.

Definition 1.2.2 *Let $A : \mathcal{D}(A) \subset X \to X$ be a closed linear operator.*

i) *The point spectrum of A denoted by $\sigma_p(A)$ is defined by*

$$\sigma_p(A) \;=\; \big\{\lambda \in \mathbb{C} :\; \lambda I - A \textit{ is not injective}\big\}.$$

Moreover each element λ of $\sigma_p(A)$ is called an eigenvalue of A and each $v \in \mathcal{D}(A)\backslash\{0\}$ satisfying $(\lambda I - A)v = 0$ is an eigenvector of A corresponding to λ.

ii) *The residual spectrum of A denoted by $\sigma_r(A)$ is defined by:*

$$\sigma_r(A) \;=\; \big\{\lambda \in \mathbb{C} :\; \lambda I - A \textit{ is injective but } \mathcal{R}(\lambda I - A) \textit{ is not dense in } X\big\}.$$

iii) *The continuous spectrum of A denoted by $\sigma_c(A)$ is defined by:*

$$\sigma_c(A) \;=\; \big\{\lambda \in \mathbb{C} :\; \lambda I - A \textit{ is injective and } \mathcal{R}(\lambda I - A) \textit{ is a proper dense subset of } X\big\}.$$

iv) *Besides, one also defines the approximate point spectrum of A denoted by $\sigma_a(A)$ by:*

$$\sigma_a(A) \;=\; \big\{\lambda \in \mathbb{C} :\; \exists (x_n)_{n\in\mathbb{N}} \subset \mathcal{D}(A) \textit{ such that } \;\; \|x_n\| = 1, \textit{ and } \lim_{n\to\infty} \|Ax_n - \lambda x_n\| = 0\big\}\,.$$

Theorem 1.2.3 *For a closed operator $A : \mathcal{D}(A) \subset X \to X$, one has*

$$\sigma_a(A) \;=\; \{\lambda \in \mathbb{C} :\; \lambda I - A \textit{ is not injective or } \mathcal{R}(\lambda I - A) \textit{ is not closed in } X\}\,.$$

Theorem 1.2.4 *Let $A : \mathcal{D}(A) \subset X \to X$ be a self-adjoint linear operator which is densely defined. Then*

$$\lambda \in \rho(A) \qquad \Longleftrightarrow \qquad \exists\, c > 0 \;:\; \|(A - \lambda I)x\,\| \geq c\|x\| \quad \forall\, x \in \mathcal{D}(A).$$

1.2.2 Semigroups of Linear Operators

In this section we will look at the semigroups of linear operators in a Banach space $\mathbf{X}$, that will be important in the project.

Definitions and properties

Definition 1.2.3 *Let $\mathbf{X}$ be a Banach space. A family $(T(t))_{t\geq 0}$ of bounded linear operators from $\mathbf{X}$ to $\mathbf{X}$ is called a strongly continuous semigroup of bounded linear operators if the following three conditions are satisfied :*

i) $T(0) = I = id_{\mathbf{X}}$

ii) $T(t+s) = T(t)T(s), \quad \forall t, s \geq 0$

iii) $\forall x \in \mathbf{X}$, *the map* $t \mapsto T(t)x \in \mathbf{X}$ *defined from* $[0, +\infty)$ *into* $\mathbf{X}$ *is continuous at the right of* 0.

A strongly continuous semigroup of bounded linear operators on $\mathbf{X}$ will be called a $\mathcal{C}_o$ semigroup.

If $\lim_{t\to 0^+} \|T(t) - I\| = 0$, then we say that the *semigroup is uniformly continuous.*

1.2.3 Examples of Semigroups

Example 1.2.4

Let's define

$$\begin{array}{cccc} T: & [0,\infty) & \longrightarrow & \mathcal{B}\big(\mathcal{C}_0(\mathbb{R})\big) \\ & t & \mapsto & T(t) \end{array}$$

which at each $f \in \mathcal{C}_0(\mathbb{R})$ assigns $T(t)f \in \mathcal{C}_0(\mathbb{R})$ defined by

$$T(t)f \;:\; x \;\mapsto\; f(t+x)\ .$$

Then the family of operators $(T(t))_{t\geq 0}$ is a strongly continuous semigroup.
Justification:
Clearly T is well-defined and (i) For all $f \in \mathcal{C}_0(\mathbb{R}$ and for all $x \in \mathbb{R}$, we have

$$(T(0)f)(x) \;=\; f(0+x) \;=\; f(x).$$

Thus $T(0)f = f$ and so $T(0) = I$
(ii) Let s, $t \in [0,\infty)$, $f \in \mathcal{C}_0(\mathbb{R})$ and $x \in \mathbb{R}$. Then we have that

$$\begin{aligned} \big((T(s)T(t))f\big)(x) &= \Big[T(s)\big(T(t)f\big)\Big](x) \\ &= \big(T(t)f\big)(s+x) \\ &= f(t+s+x) \\ &= f(s+t+x) \\ &= \big(T(s+t)f\big)(x) \end{aligned}$$

which shows that $T(s)T(t) = T(s+t)$.
(iii) We now show the continuity of $t \mapsto T(t)f$ at the right of 0. Let $f \in \mathcal{C}_0(\mathbb{R})$, then

$$\begin{aligned} \|T(t)f - f\|_{\mathcal{C}_0(\mathbb{R})} &= \sup_{x\in\mathbb{R}} \big|\big(T(t)f\big)(x) - f(x)\big| \\ &= \sup_{x\in\mathbb{R}} |f(t+x) - f(x)| \quad \longrightarrow \quad 0 \quad \text{as} \quad t \longrightarrow 0^+ \end{aligned}$$

by uniform continuity of f.
Thus

$$\lim_{t\longrightarrow 0^+} \|T(s)f - f\|_{\mathcal{C}_0(\mathbb{R})} = 0.$$

And therefore $(T(s))_{s\geq 0}$ defined above is a strongly continuous semigroup.

Example 1.2.5

Consider the following heat equation with boundary conditions

$$\left\{\begin{array}{rcll} y_t - y_{xx} & = & 0, & 0 < x < L \textit{ and } t > 0, \\ y(0,t) & = & y(L,t) = 0, & t > 0 \\ y(x,0) & = & g(x), & 0 < x < L \end{array}\right. \tag{1.22}$$

where $L > 0$, and $g \in H_0^1(0,L)$.
Using the Fourier sine expansion of $y(\cdot\,,t)$ in $L^2(0,L)$, we can apply suitably the separation of variables method which leads to the weak solution

$$y(x,t) \;=\; \sum_{k=1}^{\infty} a_k e^{-\frac{k^2\pi^2}{L^2}t}\sin(\frac{k\pi}{L}x)$$

where

$$a_k \;=\; \frac{2}{L}\int_0^L g(x)\sin(\frac{k\pi}{L}x)dx$$

are the Fourier sine coefficients of the initial value g of the problem.
Observe that for $t > 0$ fixed, $y(\cdot\,,t) \in L^2(0,L)$ with

$$||y(\cdot\,,t)||_2^2 \;\leq\; \sum_{k=1}^{\infty} |a_k|^2 \;=\; ||g||_2^2\,.$$

Also for any $\tau > 0$,

$$\sum_{k=1}^{\infty} a_k e^{-\frac{k^2\pi^2}{L^2}t}\sin(\frac{k\pi}{L}x)$$

converges absolutely on $[0,L]$ and uniformly with respect to $t \geq \tau$ since

$$|a_k|\, e^{-\frac{k^2\pi^2}{L^2}t} \;\leq\; ||g||_2\, e^{-\frac{k\pi^2}{L^2}\tau} \qquad \forall\, k \geq 1\,.$$

Note also that the function $y(\cdot\,,)$ is infinitely many times differentiable with respect to the two variables (x,t) for $t > 0$. Now define $\phi_k : [0,L] \to \mathbb{R}$ by

$$\phi_k(x) \;=\; \sqrt{\frac{2}{L}}\sin(\frac{k\pi}{L}x)\,.$$

Then $(\phi_k)_k$ forms a complete orthonormal set of $L^2(0,L)$ endowed with the inner product

$$\langle f,h\rangle \;=\; \int_0^L f(x)h(x)dx\,.$$

So we can write the solution y as

$$y(x,t) \;=\; \sum_{k=1}^{\infty} < g, \phi_k > e^{-\frac{k^2\pi^2}{L^2}t}\phi_k(x)$$

and this solution $y(\cdot\,,t)$ can be regarded as the image of the initial data g through some operator which gives rise to a semigroup.
Indeed, let us consider for each $t \geq 0$, the operator $S(t)$ defined by

$$S(t)g \;=\; \sum_{k=1}^{\infty} < g, \phi_k > e^{-\frac{k^2\pi^2}{L^2}t}\phi_k(x)\,.$$

We prove that $\big(S(t)\big)_{t\geq 0}$ is a strongly continuous semigroup of bounded operators of $L^2(0,L)$.
Firstly we show that

$$S(t) \in \mathcal{B}(L^2(0,L)), \quad \forall t \geq 0\,.$$

Clearly for each $t \geq 0$, $S(t)$ is well-defined from $L^2(0,L)$ into $L^2(0,L)$ and

$$||S(t)g||_2^2 \;\leq\; \sum_{k=1}^{\infty} |\langle g,\, \phi_k\rangle|^2 \;=\; ||g||_2^2 \qquad \forall\, g \in L^2(0,L).$$

Moreover $S(t)$ is linear because for all $g, f \in L^2(0,L)$ and all $\alpha,\ \beta\ \in \mathbb{R}$, we have

$$\begin{aligned}
S(t)(\alpha f + \beta g) \;&=\; \sum_{k=1}^{\infty}\langle \alpha + \beta g, \phi_k\rangle e^{-\frac{k^2\pi^2}{L^2}t}\phi_k(x) \\
&=\; \alpha\sum_{k=1}^{\infty}\langle f, \phi_k\rangle e^{-\frac{k^2\pi^2}{L^2}t}\phi_k(x) \;+\; \beta\sum_{k=1}^{\infty}\langle g, \phi_k\rangle e^{-\frac{k^2\pi^2}{L^2}t}\phi_k(x) \\
&=\; \alpha S(t)f \;+ \beta S(t)g\,.
\end{aligned}$$

It follows that

$$S(t) \in \mathcal{B}(L^2(0,L)), \quad \textit{and} \quad ||S(t)|| \;\leq\; 1, \qquad \forall t \geq 0\,.$$

Moreover, it is not hard to see that the Fourier series expansion of g can be obtained by considering the Fourier series expansion of the odd function defined on $[-\pi,\pi]$ by:

$$f(x) \;=\; \begin{cases} -g(-x), & x \in (-\pi, 0) \\ g(x), & x \in (0,\pi) \end{cases}$$

It then follows that,

i)

$$\begin{aligned}
S(0)g \;&=\; \sum_{k=1}^{\infty} < g, \phi_k > \phi_k(x) \\
&=\; \sum_{k=1}^{\infty}\left[\frac{2}{L}\int_0^L g(y)\sin(\frac{k\pi}{L}y)dy\right]\sin(\frac{k\pi}{L}x) \\
&=\; \sum_{k=1}^{\infty} a_k \sin(\frac{k\pi}{L}x) \\
&=\; g
\end{aligned}$$

Thus $S(0) = I$.

ii)

$$
\begin{aligned}
[S(t) \circ S(s)g](x) &= S(t)\Big(S(s)g\Big)(x) \\
&= \sum_{k=1}^{\infty} < S(s)g, \phi_k > e^{-\frac{k^2\pi^2}{L^2}t}\phi_k(x) \\
&= \sum_{k=1}^{\infty} \Big\langle \sum_{n=1}^{\infty} < g, \phi_n > e^{-\frac{k^2\pi^2}{L^2}s}\phi_n(x), \phi_k \Big\rangle e^{-\frac{k^2\pi^2}{L^2}t}\phi_(x) \\
&= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} e^{-\frac{k^2\pi^2}{L^2}s} < g, \phi_n >< \phi_n, \phi_k > e^{-\frac{k^2\pi^2}{L^2}t}\phi_(x) \\
&= \sum_{k=1}^{\infty} < g, \phi_k > e^{-\frac{k^2\pi^2}{L^2}s}e^{-\frac{k^2\pi^2}{L^2}t}\phi_k(x)
\end{aligned}
$$

since the $\phi_k' s$ form an orthogonal basis

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} < g, \phi_k > e^{-\frac{k^2\pi^2}{L^2}(t+s)}\phi_k(x) \\
&= [S(t+s)g](x)
\end{aligned}
$$

Thus $S(t+s)g = S(t) \circ S(s)g$

iii) Let $g \in\in L^2(0, L)$ be arbitrarily fixed and set $a_k = \langle g, \phi_k\rangle$ for every $k \geq 1$. Then

$$
\begin{aligned}
\|S(t)g - g\|_2^2 &= \Big\| \sum_{k=1}^{\infty} a_k e^{-\frac{k^2\pi^2}{L^2}t}\phi_k(x) - g\Big\|_2^2 \\
&= \Big\| \sum_{k=1}^{\infty} a_k e^{-\frac{k^2\pi^2}{L^2}t}\phi_k(x) - \sum_{k=1}^{\infty} a_k\phi_k(x)\Big\|_2^2 \\
&\leq \sum_{k=1}^{\infty} \Big\| a_k\phi_k(x)(e^{-\frac{k^2\pi^2}{L^2}t} - 1)\Big\|_2^2 \\
&\leq \sum_{k=1}^{\infty} |a_k|^2 \left(1 - e^{-\frac{k^2\pi^2}{L^2}t}\right)^2 \longrightarrow 0 \quad in\ l_2(\mathbb{R})\ as \quad t \longrightarrow 0^+
\end{aligned}
$$

by the dominated convergence theorem.

Thus

$$\lim_{t\longrightarrow 0^+} S(t)g \ = \ g\,.$$

Hence $\{S(t)\}_{t\geq 0}$ is a semigroup.

1.2.4 Infinitesimal Generator of a C_0-semigroup

Definition 1.2.6 *The infinitesimal generator of a $\mathcal{C}_0$ semigroup $(T(t))_{t\geq 0}$ of bounded linear operators of a Banach space $\mathbf{X}$, is the linear operator $A : \mathcal{D}(A) \longrightarrow X$ with domain*

$$\mathcal{D}(A) \;=\; \left\{ x \in \mathbf{X} : \lim_{t\to 0^+} \frac{T(t)x - x}{t} \quad exists \right\}$$

and defined by

$$Ax \;:=\; \lim_{t\to 0^+} \frac{T(t)x - x}{t} \;=\; \frac{d^+T(t)x}{dt}|_{t=0} \qquad for \quad x \in \mathcal{D}(A)\,.$$

Theorem 1.2.5 *Let $(T(t))_{t\geq 0}$ be a $\mathcal{C}_0$ semigroup. Then there exist constants $w \in \mathbb{R}$ and $M \geq 1$, such that*

$$\|T(t)\| \;\leq\; Me^{wt}\,, \qquad for \;\; 0 \leq t < +\infty\,.$$

Theorem 1.2.6 *If $(T(t))_{t\geq 0}$ is a $\mathcal{C}_0$ semigroup, then $\forall x \in \mathbf{X}$, the map $t \mapsto T(t)x$ is continuous from $\mathbb{R}^+$ into $\mathbf{X}$.*

We now give some properties of $\mathcal{C}_0$-semigroups in the following theorem.

Theorem 1.2.7 *Let $(T(t))_{t\geq 0}$ be a $\mathcal{C}_0$ semigroup and A be its infinitesimal generator. Then*

1) *For $x \in \mathbf{X}$,*

$$\lim_{h\to 0^+} \frac{1}{h} \int_t^h T(s)x\,ds \;=\; x$$

and

$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} T(s)x\,ds \;=\; T(t)x \qquad \forall\, t > 0\,.$$

2) *For $x \in \mathbf{X}$ and $t > 0$,*

$$\int_0^t T(s)x\,ds \,\in\, \mathcal{D}(A) \quad and \quad A\left(\int_0^t T(s)x\,ds\right) \;=\; T(t)x \,-\, x\,.$$

3) *For every $x \in \mathcal{D}(A)$, $T(t)x \in \mathcal{D}(A)$ and*

$$\frac{d}{dt}T(t)x \;=\; AT(t)x \;=\; T(t)Ax\,.$$

4) *For $x \in \mathcal{D}(A)$ and $s,\, t \geq 0$,*

$$T(t)x - T(s)x \;=\; \int_s^t T(\tau)Ax\,d\tau \;=\; \int_s^t AT(\tau)x\,d\tau\,.$$

Corollary 1.2.2 *If A is the infinitesimal generator of a $\mathcal{C}_0$-semigroup $(T(t))_{t\geq 0}$, then the domain $\mathcal{D}(A)$ of A is dense in $\mathbf{X}$ and A is a closed linear operator.*

In the following theorem we show that two $\mathcal{C}_0$-semigroups are equal if and only if they are generated by the same linear operator.

Theorem 1.2.8 *Let $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ be two $\mathcal{C}_0$ semigroups on $\mathbf{X}$ generated respectively by A and B. If $A = B$, then $T(t) = S(t)$, $\forall t \geq 0$.*

Proof:
Assume $A = B$ and let $x \in \mathcal{D}(A) = \mathcal{D}(B)$. Let us set

$$\alpha(s) \;=\; T(t-s)S(s)x\,, \quad s \in [0,t].$$

From Theorem 1.2.7- (3), α is differentiable and we have that:

$$\frac{d\alpha}{ds}(s) \;=\; -T(t-s)AS(s)x \,+\, T(t-s)BS(s)x \;= 0$$

since $A = B$. So $\alpha(s) = const.$ and therefore $\alpha(0) = \alpha(t)$ i.e

$$T(t)x \;=\; S(t)x, \quad \forall x \in \mathcal{D}(A).$$

By Corollary 1.2.2, $\mathcal{D}(A)$ is dense in $\mathbf{X}$ and $T(t)$, $S(t)$ are closed; therefore

$$T(t)x = S(t)x, \quad \forall x \in \mathbf{X}$$

Not all semigroups are uniformly continuous. Therefore having a linear differential equation

$$\frac{du}{dt} \;=\; Au$$

in an abstract space, it is important to find out whether the operator A is the infinitesimal generator of some semigroup. An answer is first given in the following theorem which is a characterization of the infinitesimal generator of a uniformly continuous semigroup.

Theorem 1.2.9 *A linear operator A is the infinitesimal generator of a uniformly continuous semigroup if and only if A is a bounded linear operator.*

Proof:
a) Let A be a bounded linear operator. It is well known that the series $\sum_{n=0}^{+\infty}\frac{(tA)^n}{n!}$ converges in norm for all $t \geq 0$, and defines for each such t a bounded linear operator $T(t)$.
It is not hard to see that

- $T(0) = I$
- $T(t+s) = T(t)T(s)$, for all $t, s \geq 0$.
- $e^{tA} = I + \sum_{n=1}^{+\infty}\frac{(tA)^n}{n!}$

$e^{tA} - I = tA\sum_{n=1}^{+\infty}\frac{(tA)^{n-1}}{n(n-1)!}$

Taking norm on both sides, we have

$$\begin{aligned}
\|e^{tA} - I\| &= \left\| tA\sum_{n=1}^{+\infty}\frac{(tA)^{n-1}}{n(n-1)!}\right\| \\
&\leq \|tA\|\left\|\sum_{n=1}^{+\infty}\frac{(tA)^{n-1}}{n(n-1)!}\right\| \\
&\leq \|tA\|\left\|\sum_{n=1}^{+\infty}\frac{(tA)^{n-1}}{(n-1)!}\right\| \\
&\leq t\|A\|e^{t\|A\|}
\end{aligned}$$

That is $\|T(t) - I\| \leq t\|A\|e^{t\|A\|}$, which goes to zero as t goes to 0. Now, we claim that A is the infinitesimal generator of $T(t)$. We prove this claim. Let $t > 0$. We have that
$e^{tA} - I = tA \sum_{n=1}^{+\infty} \frac{(tA)^{n-1}}{n(n-1)!}$

$\frac{e^{tA}-I}{t} = A \sum_{n=1}^{+\infty} \frac{(tA)^{n-1}}{n(n-1)!}$

$\frac{e^{tA}-I}{t} - A = A \left[\sum_{n=1}^{+\infty} \frac{(tA)^{n-1}}{n(n-1)!} - I\right]$

Taking norm on both sides we have,

$$\begin{aligned} \|\frac{e^{tA} - I}{t} - A\| &\leq \|A\| \left\|\sum_{n=1}^{+\infty} \frac{(tA)^{n-1}}{n(n-1)!} - I\right\| \\ &\leq \|A\| \left\|\sum_{n=1}^{+\infty} \frac{(tA)^{n-1}}{(n-1)!} - I\right\| \\ &= \|A\|\|e^{tA} - I\| \end{aligned}$$

That is $\left\|\frac{T(t)-I}{t} - A\right\| \leq \|A\|\|T(t) - I\|$, which implies as $t \to 0^+$ that $\frac{T(t)-I}{t} \to A$.

Thus we have established that $(T(t))_{t\geq 0}$ is a uniformly continuous semigroup of bounded linear operators generated by A.

b) Let $T(t)$ be a $\mathcal{C}_0$-semigroup of bounded linear operators on X.
Fix $\rho > 0$ small enough such that $\left\|I - \rho^{-1} \int_0^\rho T(s)ds\right\| \leq 1$. This implies that $\rho^{-1} \int_0^\rho T(s)ds$ is invertible and so also is $\int_0^\rho T(s)ds$.

Now let $h > 0$
$h^{-1}(T(h) - I) \int_0^\rho T(s)ds = h^{-1} \left(\int_0^\rho T(h+s)ds - \int_0^\rho T(s)ds\right) = h^{-1} \left(\int_\rho^{\rho+h} T(s)ds - \int_0^\rho T(s)ds\right)$
And therefore, we have
$h^{-1}(T(h) - I) = h^{-1} \left(\int_\rho^{\rho+h} T(s)ds - \int_0^\rho T(s)ds\right) \left(\int_0^\rho T(s)ds\right)^{-1}$ and letting $h \to 0$, we obtain that $h^{-1}(T(h) - I)$ converges in norm to a bounded linear operator $(T(\rho) - I) \left(\int_0^\rho T(s)ds\right)^{-1}$ which is the infinitesimal generator of $T(t)$.

Definition 1.2.7 *$(T(t))_{t\geq 0}$ is a $\mathcal{C}_0$ semigroup of contraction if and only if $\|T(t)\| \leq 1, \forall t \geq 0$*

In Theorem 1.2.9, we gave the necessary and sufficient condition for a linear operator A to be the infinitesimal generator of a uniformly continuous semigroup. The following theorem is one of the most important results in the theory of semigroups. It gives a necessary and sufficient condition for an unbounded linear operator to to be the infinitesimal generator of a $\mathcal{C}_0$ semigroup. Examples of such operators are encountered in the theory of PDEs such as the Laplacian operator.

Theorem 1.2.10 (Hille-Yosida) *A linear (unbounded) operator A is the infinitesimal generator of a $\mathcal{C}_0$-semigroup of contractions if and only if:*

i) *A is closed and $\overline{\mathcal{D}(A)} = \mathbf{X}$.*

ii) *The resolvent set of A; $\rho(A)$, contains $\mathbb{R}^+$,* *i.e*

$$\rho(A) \supset (0,+\infty),$$

and for every $\lambda > 0$

$$\|R(\lambda, A)\| \leq \frac{1}{\lambda}; \quad where \quad R(\lambda, A) = (\lambda I - A)^{-1}.$$

In an attemp to prove the above theorem, one would need the following lemmas.

Lemma 1.2.1 *Let A be a linear operator satisfying condition (i) and (ii) of Theorem 1.2.10 and*

$$R(\lambda, A) = (\lambda I - A)^{-1}.$$

Then

$$\lim_{\lambda \to +\infty} \lambda R(\lambda, A)x = x, \quad \forall x \in \mathbf{X}.$$

Proof:
Suppose first that $x \in \mathcal{D}(A)$.
One can write $R(\lambda, A)(\lambda I - A)x = x$ which implies that $\lambda R(\lambda, A)x - x = R(\lambda, A)Ax$. It follows that

$$\|\lambda R(\lambda, A)x - x\| = \|R(\lambda, A)Ax\| \leq \frac{1}{\lambda}\|Ax\|$$

by (ii) of Theorem 1.2.10. So

$$\lim_{\lambda \to +\infty} \lambda R(\lambda, A)x = x, \quad \forall x \in \mathcal{D}(A).$$

But $\mathcal{D}(A)$ is dense in $\mathbf{X}$ and $\|\lambda R(\lambda, A)\| \leq 1$ for all $\lambda > 0$. Thus for each $x \in X$ and every $z \in \mathcal{D}(A)$ we have

$$\begin{aligned} \|\lambda R(\lambda, A)x - x\| &\leq \|\lambda R(\lambda, A)(x-z)\| + \|\lambda R(\lambda, A)z - z\| + \|z - x\| \\ &\leq 2\|z - x\| + \|\lambda R(\lambda, A)z - z\|. \end{aligned}$$

Therefore

$$\lambda R(\lambda, A)x \to x \quad as \quad \lambda \to +\infty$$

for every $x \in \mathbf{X}$.

We now define for every $\lambda > 0$, the Yosida approximation of A by:

$$A_\lambda = \lambda A R(\lambda, A) = \lambda^2 R(\lambda, A) - \lambda I.$$

Lemma 1.2.2 *Let A be a linear operator satisfying the conditions (i) and (ii) of Theorem 1.2.10 and denote by A_λ the Yosida approximation of A. Then*

$$\lim_{\lambda \to +\infty} A_\lambda x = Ax, \quad \forall x \in \mathcal{D}(A).$$

Lemma 1.2.3 *Let A satisfy conditions (i) and (ii) of Theorem 1.2.10 and denote by A_λ the Yosida approximation of A. Then A_λ is the infinitesimal generator of a $\mathcal{C}_0$-semigroup of contractions $\left(e^{tA_\lambda}\right)_{t\geq 0}$. Furthermore, for every $x \in \mathbf{X}$, λ, $\mu > 0$, we have*

$$\|e^{tA_\lambda}x - e^{tA_\mu}x\| \ \leq\ t\|A_\lambda x - A_\mu x\|, \quad t \geq 0\,.$$

Proof:
From the definition of A_λ, we have

$$A_\lambda \ = \ \lambda^2 R(\lambda, A) - \lambda I$$

so that A_λ is bounded as the sum of two bounded linear operators. Moreover

$$\|A_\lambda\| \ \leq\ \|\lambda^2 R(\lambda, A)\| \,+\, \|\lambda I\| \ \leq\ 2\lambda\,.$$

So A_λ is a bounded linear operator and therefore is the generator of a $\mathcal{C}_0$ semigroup $\left(e^{tA_\lambda}\right)_{t\geq 0}$ of bounded linear operators.
Claim: $\|e^{tA_\lambda}\| \ \leq\ 1, \quad \forall\, t \geq 0$
Proof of the claim

$$\begin{aligned} e^{tA_\lambda} \ &= \ e^{(\lambda^2 tR(\lambda,A) - \lambda t Id_{\mathbf{X}})} \\ &= \ e^{\lambda^2 tR(\lambda,A)}.e^{-\lambda t Id_{\mathbf{X}}} \end{aligned}$$

Taking the norm one gets,

$$\begin{aligned} \|e^{tA_\lambda}\| \ &= \ e^{-\lambda t}\|e^{\lambda^2 tR(\lambda,A)}\| \\ &\leq \ e^{-\lambda t}e^{\|\lambda^2 tR(\lambda,A)\|}, \quad \textit{since} \quad e^{\|\lambda^2 tR(\lambda,A)\|} \leq e^{\lambda t} \\ &\leq \ 1 \end{aligned}$$

And therefore e^{tA_λ} is a $\mathcal{C}_0$ semigroup of contractions, also from the definition we see that e^{tA_λ}, A_λ, e^{tA_μ} and A_μ commute pairwise.
Now for $x \in X$ and $t \geq 0$ fixed, let

$$f(s) \ = \ e^{t(1-s)A_\lambda}e^{stA_\mu}x\,, \quad 0 \leq s \leq 1$$

so that

$$f(0) \ = \ e^{tA_\lambda}x \quad \text{and} \quad f(1) \ = \ e^{tA_\mu}x\,.$$

Then f is differentiable on $[0, 1]$ and

$$\begin{aligned} f'(s) \ &= \ -tA_\lambda e^{t(1-s)A_\lambda}e^{stA_\mu}x \ + \ tA_\mu e^{t(1-s)A_\lambda}e^{stA_\mu}x \\ &= \ te^{t(1-s)A_\lambda}e^{stA_\mu}(A_\mu x \ - \ A_\lambda x) \end{aligned}$$

satisfies

$$||f'(s)|| \ \leq\ t||A_\mu x \ - \ A_\lambda x|| \ \ \forall\, s \in [0, 1]\,.$$

Therefore by the Mean Value Theorem we have

$$||f(0) - f(1)|| \ \leq \ \sup_{0<s<1} ||f'(s)||\,;$$

that is

$$\|e^{tA_\lambda}x \ - \ e^{tA_\mu}x\| \ \leq\ t\|A_\lambda x - A_\mu x\|\,.$$

1.2.5 Lumer-Phillips Theorem

Let $\mathbf{X}$ be a Banach space and $\mathbf{X}^*$ be its dual. For every $x \in \mathbf{X}$, we define the duality set $J(x) \subseteq \mathbf{X}^*$ by

$$J(x) \ = \ \left\{x^* : x^* \in \mathbf{X}^* \text{and } \langle x^*, x\rangle = \|x\|^2 = \|x^*\|\right\}.$$

Definition 1.2.8 *A linear operator $A : \mathcal{D}(A) \subset \mathbf{X} \longrightarrow \mathbf{X}$ is said to be dissipative if for every $x \in \mathcal{D}(A)$,*

$$Re\langle Ax,\, x^*\rangle \ \leq \ 0 \quad \textit{for some } x^* \in J(x)\,.$$

Theorem 1.2.11 *A linear operator $A : \mathcal{D}(A) \subset \mathbf{X} \to \mathbf{X}$ is dissipative if and only if*

$$\|(\lambda I - A)x\| \ \geq \ \lambda\|x\|, \qquad \forall x \in \mathcal{D}(A) \quad \textit{and} \quad \lambda > 0\,.$$

Definition 1.2.9 *A linear operator $A : \mathcal{D}(A) \subset \mathbf{X} \to \mathbf{X}$ is said to be m-dissipative if A is dissipative and $\lambda I - A$ is surjective for every $\lambda > 0$.*

Remark 1.2.2 [2] *Let $A : \mathcal{D}(A) \subset \mathbf{X} \to \mathbf{X}$ be an operator of a Banach space X.*

i) *A is m-dissipative if A is dissipative and there exists $\lambda_0 > 0$ such that $\lambda_0 I - A$ is surjective.*

ii) *If A is m-dissipative, then A is closed.*

iii) *If X is a Hilbert space and A is m-dissipative, then A is closed and its domain is dense in X.*

Theorem 1.2.12 (Lumer- Phillips) *Let A be a densely defined operator on some Banach space $\mathbf{X}$.*

a) *If A is m-dissipative (i.e., A is dissipative and there exists $\lambda_0 > 0$ such that $Im(\lambda_0 I - A) = \mathbf{X}$), then A is the infinitesimal generator of a $\mathcal{C}_0$-semigroup of contractions on $\mathbf{X}$.*

b) *If A is the infinitesimal generator of a $\mathcal{C}_0$-semigroup $\big(T(t)\big)_{t\geq 0}$ of contractions on $\mathbf{X}$ then*

$$Im(\lambda I - A) \ = \ \mathbf{X} \quad \forall\, \lambda > 0$$

and A is dissipative. Moreover

$$\forall\, x \in \mathcal{D}(A) \quad \textit{and} \quad x^* \in J(x), \quad Re < Ax, x^* > \leq 0.$$

Also in this case $(0, +\infty) \subset \rho(A)$ and for all $\lambda > 0$ and $\forall\, x \in \mathbf{X}$

$$R(\lambda, A)x \ := \ (\lambda I - A)^{-1}x \ = \ \int_0^{+\infty} e^{-\lambda s}T(s)x\, ds\,.$$

Example 1.2.10

Let

$$X = \{u \in \mathcal{C}([0.1]) \,:\, u(0) = 0\}$$

equipped with the sup-norm:

$$\|u\|_\infty = \max_{0\le x\le 1} |u(x)|\,, \tag{1.23}$$

and consider the linear (differential) operator

$$A = -\tfrac{d}{dx}$$

with domain

$$\mathcal{D}(A) = \{u \in \mathcal{C}^1([0,1]),\ u(0) = u'(0) = 0\}. \tag{1.24}$$

Then X is a Banach space as a closed subspace of $(\mathcal{C}([0.1])$ and also one easily shows that $\mathcal{D}(A)$ is dense in X.

We prove that A is m-dissipative; i.e., $I - \lambda A$ is bijective and has a bounded inverse satisfying $\|(I-\lambda A)^{-1}\| \le 1$. To do this, it suffices to verify that for any $f \in X$, the problem

$$\begin{cases} \lambda\frac{du}{dx} + u &= f, \\ u(0) &= u_0 \end{cases} \tag{1.25}$$

has a unique solution u in $\mathcal{D}(A)$, and $\|u\|_\infty \le ||f||_\infty$.

Indeed,(1.25) has a unique solution that is explicitely expressed as

$$u(x) = \frac{1}{\lambda} e^{-\frac{1}{\lambda}x} \int_0^x e^{\frac{\xi}{\lambda}} f(\xi) d\xi \qquad 0 \le x \le 1. \tag{1.26}$$

The function u defined above is continuously differentiable. Moreover one easily sees that $u \in \mathcal{D}(A)$, and for all $x \in [0,1]$, we have

$$\begin{aligned} |u(x)| &\le \frac{1}{\lambda} e^{-\frac{1}{\lambda}x} \|f\| \int_0^x e^{\frac{\xi}{\lambda}} d\xi \\ &= (1 - e^{-\frac{1}{\lambda}x}) \|f\|_\infty \le \|f\|_\infty \end{aligned}$$

which yields $\|u\|_\infty \le \|f\|_\infty$ and we conclude that A is m-accretive.

The corresponding evolution equation is

$$\begin{cases} \frac{du}{dt} &= Au, \\ u(0) &= u_0 \end{cases} \tag{1.27}$$

where the unknown u is such that $u(t) \in X$ for $t \ge 0$, that is

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} &= 0, \\ u(t,0) &= 0, \\ u(0,x) &= u_0(x) \end{cases} \tag{1.28}$$

by writing $u(t)(x) = u(t,x)$.
Since A is m-dissipative, A is the infinitesimal generator of a semigroup of contractions $\big(T(t)\big)_{t\geq 0}$.
Therefore this evolution problem has a unique mild solution

$$u\,:\; t \mapsto u(t) = T(t)u_0$$

for every $u_0 \in X$. This solution is a classical solution when $u_0 \in \mathcal{D}(A)$.

One can really solve this problem when $u_0 \in \mathcal{D}(A)$ by making the change of variable

$$(y,s) \;=\; (t-x,\, t+x)$$

to get

$$u(x,t) \;=\; \begin{cases} u_0(x-t) & if \quad x \geq t, \\ \\ \quad 0 & if \quad 0 \leq x < t \end{cases} \tag{1.29}$$

Thus if u_0 is any continuous but not differentiable function, then for any $t \geq 0$, the function u defined above is not differentiable, and neither belongs to $\mathcal{D}(A)$.

When $u_0 \in X$, but does not belong to $\mathcal{D}(A)$, we still can view $u = S(t)u_0$ as a sort of solution in the generalized sense, and it is usually called the *mild solution* in the literature. On the other hand in some special cases when $u_0 \in X$, for $t > 0$, $S(t)u_0$ is still a classical solution. Also one can check directly that the map defined by $S(t)u_0 := u(t,u_0)$ is a semigroup.

Now we specify the definition of m-accretive operators.

Definition 1.2.11 (Dissipative operator in Hilbert spaces) *Let H be a real Hilbert space. An unbounded linear operator*
$A : \mathcal{D}(A) \subset \mathbf{H} \to \mathbf{H}$ is said to be dissipative on H if

$$< Av, v >_H \leq 0, \quad \forall v \in \mathcal{D}(A)$$

.
A is m-dissipative if A is dissipative and moreover
$\forall f \in H,\ \forall \lambda > 0 \quad \exists!\, u \in \mathcal{D}(A)$ such that $\lambda u - Au = f$,

Definition 1.2.12 *Let H be a Hilbert space. An unbounded linear operator*

$$A :\, \mathcal{D}(A) \subset \mathbf{H} \to \mathbf{H}$$

is said to be monotone (or accretive) if

$$\langle Av, v\rangle \;\geq\; 0, \quad \forall v \in \mathcal{D}(A)\,.$$

A is said to be maximal monotone if A is monotone and moreover $\mathcal{R}(I + A) = H$. That is

$$\forall f \in H,\;\; \exists\, u \in \mathcal{D}(A) \quad such\ that \quad u + Au = f\,.$$

Remark 1.2.3 *A is monotone (or accretive) if and only if $-A$ is dissipative*

Note that in a Hilbert space, an operator A is said to be maximal monotone, if $-A$ is m-dissipative.

Proposition 1.2.1 *Let A be a maximal monotone Operator in a Hilbert space H. Then*

i) *$\mathcal{D}(A)$ is dense in H*

ii) *A is closed*

iii) *For all $\lambda > 0, (I + \lambda A)$ is bijective from $\mathcal{D}(A)$ onto H.*
$(I + \lambda A)^{-1}$ is bounded and $\|(I + \lambda A)^{-1}\|_{\mathcal{L}(H)} \leq 1$

Theorem 1.2.13 (Hille-Yosida in Hilbert Spaces) *An unbounded linear operator $(A, \mathcal{D}(A))$ in H is the infinitesimal generator of a semigroup of contractions on H if and only if A is m-dissipative in H.*

Example 1.2.13 (The Laplace operator in $L^2(\Omega)$) *Let Ω be a bounded regular subset of $\mathbb{R}^N$, with a boundary Γ of class $\mathcal{C}^2$. Set*

$$H = L^2(\Omega), \qquad \mathcal{D}(A) = H^2 \cap H_0^1(\Omega), \qquad Ay = \Delta y.$$

A is dissipative, since we have the following

$$\langle Au, u \rangle_H = \int_\Omega (\Delta u) u dx = -\int_\Omega \langle \nabla u, \nabla u \rangle dx \leq 0\,.$$

A is m-dissipative because for every $\lambda > 0$ and every $f \in H$, the equation $\lambda y - \Delta y = f$ has a unique solution in $\mathcal{D}(A)$.
Therefore the Laplacian generates a $\mathcal{C}_0$-semigroup of contractions of $H_0^1(\Omega)$. This is a generalization of Example 1.2.5.

The characterization of the infinitesimal generators of general $\mathcal{C}_0$ semigroups.

Lemma 1.2.4 *Let A be any linear operator for which*

$$\rho(A) \supseteq]0, +\infty) \quad if \quad \|R(\lambda, A)^n\| \leq \frac{M}{\lambda^n}, \ \forall \lambda > 0, \quad n \geq 1\,, \, M \geq 1.$$

Then there exists a norm $|\cdot|$ on $\mathbf{X}$ which is equivalent to the original norm $\|\cdot\|$ on $\mathbf{X}$ and satisfies

$$\|x\| \leq |x| \leq M\|x\| \quad for \quad x \in \mathbf{X}, \quad and \quad |R(\lambda, A)x| \leq \frac{|x|}{\lambda}, x \in \mathbf{X}, \lambda > 0$$

Theorem 1.2.14 *A linear operator A is the infinitesimal generator of a $\mathcal{C}_0$ semigroup $(T(t))_{t \geq 0}$ satisfying $\|T(t)\| \leq M \quad (M \geq 1)$ if and only if.*
i) A is closed and $\mathcal{D}(A)$ is dense in $\mathbf{X}$.
ii) The resolvent set $\rho(A)$ contains $\mathbb{R}^+$ and

$$\|R(\lambda, A)^n\| \leq \frac{M}{\lambda^n}, \forall \lambda > 0, n \geq 1$$

Remark: The properties of $(T(t))_{t\geq 0}$ and the those of its infinitesimal generator remain unchanged to a new equivalent norm on $\mathbf{X}$.

Theorem 1.2.15 *A linear operator A is the infinitesimal generator of a $\mathcal{C}_0$ semigroup $(T(t))_{t\geq 0}$ satisfying $\|T(t)\| \leq e^{wt}$ (for some $w \geq 0$) if and only if:*

i) A is closed and $\mathcal{D}(A)$ is dense in $\mathbf{X}$.

ii) The resolvent set $\rho(A)$ contains $(w, +\infty)$ and $\|R(\lambda, A)\| \leq \frac{1}{\lambda - w}, \forall \lambda > w$

Theorem 1.2.16 (Hille-Yosida General case) *A linear operator A is the infinitesimal generator of a $\mathcal{C}_0$ semigroup $(T(t))_{t\geq 0}$ satisfying $\|T(t)\| \leq Me^{wt}$ for $t \geq 0, M \geq 1, w \in \mathbb{R}$ if and only if.*
i) A is closed and $\mathcal{D}(A)$ is dense in $\mathbf{X}$.

ii) The resolvent set $\rho(A) \supseteq (w, +\infty)$ and $\|R(\lambda, A)^n\| \leq \frac{M}{(\lambda - w)^n}, \forall \lambda > w, n \geq 1$

Having stated the important theorems and definitions that we will use , we can now move to the next chapter and start applying them to the theory we are studying.

CHAPTER 2

ABSTRACT LINEAR EVOLUTION EQUATIONS

Most often significant external forces affect the evolution of a process. Let X be a real Banach space. In this chapter, we discuss the non-homogeneous Cauchy problem

$$\begin{cases} U'(t) &= AU(t) + f(t), \quad t > 0 \\ U(0) &= U_0 \end{cases}$$

where $A : \mathcal{D}(A) \subset X \longrightarrow X$ is a given linear operator and $f : [0, \infty) \longrightarrow X$ is a given function of the time variable only. This equation is called linear evolution equation. Basically we shall study the existence and uniqueness of solutions of the above problem, imposing different conditions on f.

2.1 Linear Evolution Equations in finite dimensional spaces: Well-Posedness

In this section, we examine the linear Cauchy problem in the finite dimensional case. In this case we identify the linear operator A with an $N \times N$ matrix. We shall see that if the forcing term is continuous, then there is existence and uniqueness of the solution. So consider the following initial value problem (I.V.P) :

$$\begin{cases} U'(t) &= AU(t) + f(t), \quad t > 0 \\ U(0) &= U_0 \end{cases} \tag{2.1}$$

where A$\in \mathbb{M}^N(\mathbb{R}), f : [0, \infty) \longrightarrow \mathbb{R}^N$ and $U_0 \in \mathbb{R}^N$. We shall formulate the theory for (2.1).
We have
$U'(s) = AU(s) + f(s)$

$\Longleftrightarrow U'(s) - AU(s) = f(s),$

$\Longleftrightarrow e^{-sA}(U'(s) - AU(s)) = e^{-sA} f(s)$

$\Longleftrightarrow \frac{d}{ds}(e^{-sA}U(s)) = e^{-sA} f(s), \text{since}, \quad \frac{d}{ds}(e^{-sA}) = -Ae^{-sA}.$

$\Longleftrightarrow \int_0^t \frac{d}{ds}(e^{-sA}U(s))ds = \int_0^t e^{-sA}f(s)ds$

$\Longleftrightarrow e^{-tA}U(t) - U(0) = \int_0^t e^{-sA}f(s)ds$

It implies that

$$U(t) = e^{tA}U_0 + \int_0^t e^{(t-s)A}f(s)ds \tag{2.2}$$

Definition 2.1.1 *Let* $U : [0,T] \longrightarrow \mathbb{R}^N$ *be a function*

a) *U is a classical solution of (2.1) if*

i) U is continuous on $[0,T]$
ii) U is differentiable on $(0,T]$
iii) U satisfies (2.1)

b) *U is a mild solution of (2.1) if*

i) U is continuous on $[0,T]$
ii) U is given by (2.2), $\forall t \in [0,T]$.

Theorem 2.1.1 (Existence and Uniqueness) *Let* $T > 0$ *and suppose that* $f \in \mathcal{C}([0,T];\mathbb{R}^N)$. *Then (2.1) has a unique classical solution on* $[0,T]$ *given by (2.2).*

Proof:
Existence

Let U be given by (2.2). From the continuity of f we have that the map $t \mapsto \int_0^t e^{(t-s)A}f(s)ds$ and also the map $t \mapsto e^{tA}U_0$ is continuous. therefore U given by (2.2) is continuous as the sum of two continuous functions. Moreover

$$\begin{aligned} U'(t) &= Ae^{tA}U_0 + f(t) + A\int_0^t e^{(t-s)A}f(s)ds \\ &= A(e^{tA}U_0 + \int_0^t e^{(t-s)A}f(s)ds) + f(t) \\ &= AU(t) + f(t) \end{aligned}$$

Also $U(0) = e^{0A}U_0 = e^0U_0 = U_0$
Thus U is a classical solution of (2.2).
Uniqueness:
Suppose that U and V are both classical solutions of (2.1). Then define $Z : [0,T] \longrightarrow \mathbb{R}^N$ by $Z(t) = U(t) - V(t)$ .Then Z is continuous and differentiable on [0,T] and (0,T] respectively as the sum of two continuous and differentiable functions. Moreover

$$\begin{aligned} Z'(t) &= U'(t) - V'(t) \\ &= A(U(t) - V(t)) \\ &= AZ(t) \end{aligned}$$

So $Z(t) = e^{tA}Z_0$ but $Z_0 = Z(0) = 0$, thus $Z(t) = 0, \forall t \in [0,T]$ and therefore $U = V$ proving uniqueness and completing the proof of the theorem.

Continuous dependence on the given data:
Consider the following perturbed system from (2.1).

$$\begin{cases} V'(t) &= AV(t) + g(t), \quad t > 0 \\ V(0) &= V_0 \end{cases} \tag{2.3}$$

where A is the matrix given in (2.1).

We are hopeful that the difference between the solutions U and V of (2.1) and (2.3), respectively, in the sense of the *supnorm* on $\mathcal{C}([0,T];\mathbb{R}^N)$, for any time interval [0,T] can be controlled by making the error terms sufficiently small. In this case we say that the solution depends continuously on the given data. We summarize this in the following proposition.

Proposition 2.1.1 *Let $\epsilon_1, \epsilon_2, T > 0$. Assume that $f, g \in \mathbf{L}^1(0,T;\mathbb{R}^N)$ and $U_0, V_0, \in \mathbb{R}^N$. If $\|f-g\|_{\mathbf{L}^1(0,T;\mathbb{R}^N)} < \epsilon_1$ and $\|U_0 - V_0\|_{\mathbb{R}^N} < \epsilon_2$, then*

$$\|U - V\|_{\mathcal{C}([0,T];\mathbb{R}^N)} \leq e^{T\|A\|_{\mathbb{M}^N(\mathbb{R})}}(\epsilon_1 + \epsilon_2),$$

where U and V are the mild solutions of (2.1) and (2.3) respectively.

Proof: We have that

$$\begin{aligned} \|U(t) - V(t)\| &\leq \|e^{tA}\|\|U_0 - V_0\| + \left\| \int_0^t e^{(t-s)A}[f(s) - g(s)]ds \right\| \\ &\leq \|e^{tA}\|\|U_0 - V_0\| + \int_0^t \|e^{(t-s)A}\|\|f(s) - g(s)\|_{\mathbb{R}^N} ds \\ &\leq e^{T\|A\|}\|U_0 - V_0\| + e^{t\|A\|} \int_0^T \|f(s) - g(s)\|_{\mathbb{R}^N} ds \\ &\leq e^{t\|A\|}\|U_0 - V_0\|_{\mathbb{R}^N} + e^{t\|A\|}\|f - g\|_{\mathbf{L}^1(0,T;\mathbb{R}^N)} \\ &< e^{t\|A\|}(\epsilon_1 + \epsilon_2). \end{aligned}$$

Taking the supremum over $[0,T]$ yields

$$\|U - V\|_{\mathcal{C}([0,T];\mathbb{R}^N)} \leq e^{T\|A\|_{\mathbb{M}^N}}(\epsilon_1 + \epsilon_2),$$

completing the proof of the theorem.$\Box$

We now state an important result showing that any mild solution of (2.3) can be approximated by a sequence of classical solutions.

Theorem 2.1.2 *Let $g \in \mathbf{L}^1(0,T;\mathbb{R}^N), V_0 \in \mathbb{R}^N$ and V be a mild solution of (2.3). Then there exists a sequence $\{V_n\}_{n\geq 1}$ of classical solutions of a sequence of aptly constructed I.V.Ps such that*

$$\lim_{n\to\infty} \|V_n - V\|_{\mathcal{C}([0,T];\mathbb{R}^N)} = 0$$

Proof:
We have that the set $\mathcal{C}([0,T];\mathbb{R}^N)$ is dense in $\mathbf{L}^1(0,T;\mathbb{R}^N)$.
So $g \in \mathbf{L}^1(0,T;\mathbb{R}^N)$ implies that there exists $\{g_n\}_{n\geq 1} \subset \mathcal{C}([0,T];\mathbb{R}^N)$ such that $\lim_{n\to\infty} \|g_n - g\|_{\mathbf{L}^1(0,T;\mathbb{R}^N)} = 0$.
Now consider the sequence of I.V.Ps

$$\left\{ \begin{array}{rcl} V_n'(t) &=& AV_n(t) + g_n(t), \quad t > 0 \\ V_n(0) &=& V_0 \end{array} \right. \tag{2.4}$$

In view of Theorem 2.1.1, each of these I.V.Ps has a classical solution since $\{g_n\}_{n\geq 1} \subset \mathcal{C}([0,T];\mathbb{R}^N)$. So we have a sequence $\{V_n\}_{n\geq 1}$ of classical solutions given by (2.2). Moreover we have

$$\begin{array}{rcl} \|U(t) - V(t)\| &\leq& \|\int_0^t e^{(t-s)A}[g_n(s) - g(s)]ds\| \\ &\leq& \int_0^t \|e^{(t-s)A}\| \|g_n(s) - g(s)\|_{\mathbb{R}^N} ds \\ &\leq& e^{t\|A\|_{\mathbb{M}^N}} \int_0^T \|g_n(s) - g(s)\|_{\mathbb{R}^N} ds \\ &\leq& e^{t\|A\|_{\mathbb{M}^N}} \|g_n - g\|_{\mathbf{L}^1(0,T;\mathbb{R}^N)} \end{array}$$

Taking the supremum over [0,T] we have

$$\|V_n - V\|_{\mathcal{C}([0,T];\mathbb{R}^N)} \leq e^{T\|A\|_{\mathbb{M}^N}} \|g_n - g\|_{\mathbf{L}^1(0,T;\mathbb{R}^N)}$$

and therefore

$$\lim_{n\to\infty} \|V_n - V\|_{\mathcal{C}([0,T];\mathbb{R}^N)} = 0$$

2.2 Linear Evolution Equations in infinite Dimensional Spaces: Abstract Cauchy Problem

Let X be a Banach space and A a linear operator from $\mathcal{D}(A) \subset X$ into X. Given $u_0 \in X$, the Abstract Cauchy problem for A with initial data u_0 consists of finding a solution u to the initial value problem:

$$\left\{ \begin{array}{rcl} u'(t) &=& Au(t) + f(t), \quad t > 0 \\ u(0) &=& u_0 \end{array} \right. \tag{2.5}$$

where $f : [0,\infty) \to X$ and by solution we mean a classical solution as defined above.

We shall focus our attention on the case in which A is the infinitesimal generator of a $\mathcal{C}_0$-semigroup denoted by $\{e^{tA} : t \geq 0\}$. Following the same steps as in the finite dimensional case, with slight modification we can derive the following variation of parameter formula.

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A} f(s)ds \tag{2.6}$$

We have the following result.

Theorem 2.2.1 *If $u_0 \in \mathcal{D}(A)$ and $f \in \mathcal{C}([0,T];\mathcal{D}(A))$ is such that the function $t \mapsto \int_0^t e^{(t-s)A}f(s)ds$ belongs to $\mathcal{C}^1((0,T);\mathcal{D}(A))$. Then (2.5) has a unique classical solution given by (2.6).*

Proof: We must show that u given by (2.6) satisfies the definition of a classical solution. We have that u is continuous as the sum of two continuous functions namely

$$t \mapsto e^{tA}u_0, \quad and \quad t \mapsto \int_0^t e^{(t-s)A}f(s)ds.$$

We first show that

$$u(t) \in \mathcal{D}(A), \forall t \in [0,T].$$

This requires that we verify that $\frac{e^{hA}u(t)-u(t)}{h}$ converges as $h \to 0^+$.
$u_0 \in \mathcal{D}(A)$ implies that

$$Au_0 = \lim_{h\to 0^+} \frac{e^{hA}u_0 - u_0}{h},$$

and it follows from a well-known property of semigroup (Theorem 1.2.7-3)) that $e^{tA}u_0 \in \mathcal{D}(A), \forall t \in [0,T]$.
Now since $\mathcal{D}(A)$ is a linear subspace of X, we show the existence of

$$\lim_{h\to 0^+} \frac{e^{hA}-I}{h}\Big(\int_0^t e^{(t-s)A}f(s)ds\Big)$$

and conclude.
We have that

$$A = \lim_{h\to 0^+} \frac{e^{hA}-I}{h},$$

so we can write.

$$\begin{aligned} \frac{e^{hA}-I}{h}\Big(\int_0^t e^{(t-s)A}f(s)ds\Big) &= \int_0^t \frac{e^{hA}-I}{h}\big(e^{(t-s)A}f(s)\big)ds \\ &= \int_0^t \frac{e^{hA}e^{(t-s)A}f(s) - e^{(t-s)A}f(s)}{h}ds \\ &= \Big[\frac{\int_0^{t+h} e^{(t+h-s)A}f(s)ds - \int_0^t e^{(t-s)A}f(s)ds}{h} - \frac{1}{h}\int_t^{t+h} e^{(t+h-s)A}f(s)ds\Big] \end{aligned}$$

Taking the limit on both sides as $h \to 0^+$, we obtain

$$A\int_0^t e^{(t-s)A}f(s)ds = \frac{d}{dt}\Big(\int_0^t e^{(t-s)A}f(s)ds\Big) - f(t), \qquad (1)$$

By assumption

$$\int_0^t e^{(t-s)A}f(s)ds \in \mathcal{C}^1((0,T);\mathcal{D}(A))$$

So

$$\frac{d}{dt}\Big(\int_0^t e^{(t-s)A}f(s)ds\Big) \in \mathcal{D}(A)$$

and also $f(t) \in \mathcal{D}(A)$, and therefore by linearity of $\mathcal{D}(A)$, $u(t) \in \mathcal{D}(A)$, $\forall t \in [0,T]$.
Finally we show that u given by (2.6) satisfies (2.5). Observe that (1) implies that

$$\frac{d}{dt}\Big(\int_0^t e^{(t-s)A}f(s)ds\Big) = A\int_0^t e^{(t-s)A}f(s)ds + f(t) \tag{2.7}$$

Also

$$\frac{d}{dt}(e^{tA}u_0) = Ae^{tA}u_0 \tag{2.8}$$

Using the linearity of A and adding (2.7) and (2.8) yields

$$\begin{aligned} u'(t) &= \frac{d}{dt}\Big(e^{tA}u_0 + \int_0^t e^{(t-s)A}f(s)ds\Big) \\ &= A\Big(e^{tA}u_0 + \int_0^t e^{(t-s)A}f(s)ds\Big) + f(t) \\ &= Au(t) + f(t) \end{aligned}$$

and $u(0) = u_0$. Thus u satisfies (2.5) and uniqueness also follows easily. Hence the proposition is proved.$\square$

Remark 2.2.1 *The continuity of f in general is not sufficient to guarantee the existence of a classical solution of (2.5), for $u_0 \in \mathcal{D}(A)$. As we can see in the following example.*

Example 2.2.1 *Let A be the infinitesimal generator of a $\mathcal{C}_0$-semigroup $T(t)$, and suppose there exits $y \in X$ such that $T(t)y \notin \mathcal{D}(A)$ for any $t \geq 0$.*
Let $f(s) = T(s)y$. Then f is continuous and the initial value problem

$$\begin{cases} u'(t) = Au(t) + T(t)y, \quad t > 0 \\ u(0) = 0 \end{cases} \tag{2.9}$$

has no solution even though $u(0) = 0 \in \mathcal{D}(A)$.

Proof: The mild solution of (2.9) is defined by:
$u(t) = T(t)0 + \int_0^t T(t-s)T(s)yds = \int_0^t T(t)yds = tT(t)y$
But the function $t \mapsto tT(t)y$ is not differentiable for $t > 0$ since $y \notin \mathcal{D}(A)$ and therefore cannot be the classical solution of (2.9).

We now examine the Cauchy Problems where the forcing term $f \in L^1_{loc}([0,\infty),X)$, where $L^1_{loc}([0,\infty),X)$ is the set of all measurable functions $f : [0,\infty) \longrightarrow X$ that are integrable on compact subsets of $[0,\infty)$.

Definition 2.2.2 *We assume that $f \in L^1_{loc}([0,\infty),X)$. A continous function $u : [0,\infty) \longrightarrow X$ is called a strong (or classical) solution of (2.5) if*

i) *$u(t) \in \mathcal{D}(A)$ a.e. on $[0,\infty)$.*

ii) *$Au(.) \in L^1_{loc}([0,\infty),X)$.*

iii) $u(t) = u_0 + \int_0^t Au(s)ds + \int_0^t f(s)ds, \quad t \geq 0$

Instead of (ii) and (iii) we can require the following equivalent properties:

(ii') $u(.) \in W^{1,1}_{loc}([0,\infty), X)$,

(iii') $u(0) = u_0$ *and* $\frac{d}{dt}u(t) = Au(t) + f(t)$ *a.e. on* $t \geq 0$

where

$W^{1,1}_{loc}([0,\infty), X) = \Big\{ f \in L^1_{loc}([0,\infty), X) | \, \exists g \in L^1_{loc}([0,\infty), X) \text{ with } f(t) = f(0) + \int_0^t g(s)ds; \text{for } t \geq 0 \Big\}$

We have the following theorem which is a version of *Theorem* 2.2.1, when the forcing term belongs to $L^1_{loc}([0,\infty), X)$

Theorem 2.2.2 *Let A be the infinitesimal generator of a $\mathcal{C}_0$-semigroup $S(.)$ and assume $f \in L^1_{loc}([0,\infty), X)$.*
a) Let $u_0 \in \mathcal{D}(A)$ be given and assume that $f \in W^{1,1}_{loc}([0,\infty), X)$. Then the function u defined by

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s)ds, \quad t \geq 0, \tag{2.10}$$

is a strong solution of (2.5). Moreover, we have $u \in \mathcal{C}([0,\infty); \mathcal{D}(A))$.
b) Assume that u is a strong solution of (2.5). Then u is given by (2.10).

Lemma 2.2.1 *Let $S(.)$ be $\mathcal{C}_0$-semigroup with infinitesimal generator A. Then*

i) *For any $x \in \mathcal{D}(A)$, $\quad S(.)x \in \mathcal{C}([0,\infty); \mathcal{D}(A))$*

ii) *For any $f \in W^{1,1}_{loc}([0,\infty), X)$ we have*

$$\int_0^t S(t-s)f(s)ds \in \mathcal{D}(A), \quad t \geq 0$$

$$A\int_0^t S(t-s)f(s)ds = S(t)f(0) - f(t) + \int_0^t S(t-s)f'(s)ds \tag{2.11}$$

Proof:
i) The proof of (i) is the same as the proof of $e^{tA}u_0 \in \mathcal{D}(A)$ in Theorem 2.2.1, since we can denote $S(t)$ by e^{tA}.

ii) For $h > 0$, we have that

$$\begin{aligned}
\frac{1}{h}\big(S(h)-I\big)\int_0^t S(t-s)f(s)ds &= \frac{1}{h}\int_0^t S(t+h-s)f(s)ds - \frac{1}{h}\int_0^t S(t-s)f(s)ds \\
&= \frac{1}{h}\int_0^{t+h} S(t+h-s)f(s)ds - \frac{1}{h}\int_t^{t+h} S(t+h-s)f(s)ds \\
&\quad -\frac{1}{h}\int_0^t S(t-s)f(s)ds \\
&= \frac{1}{h}\int_0^{h} S(t+h-s)f(s)ds + \frac{1}{h}\int_h^{t+h} S(t+h-s)f(s)ds \\
&\quad -\frac{1}{h}\int_t^{t+h} S(t+h-s)f(s)ds - \frac{1}{h}\int_0^t S(t-s)f(s)ds \\
&\quad \text{making a change of variable, it follows that} \\
&= \frac{1}{h}\int_0^{h} S(t+h-s)f(s)ds - \frac{1}{h}\int_t^{t+h} S(t+h-s)f(s)ds \\
&\quad + \int_0^t S(t-s)\Big[\frac{1}{h}\big(f(s+h)-f(s)\big)\Big]ds
\end{aligned}$$

Taking the limit as $h \longrightarrow 0^+$ we have that $\int_0^t S(t-s)f(s)ds \in \mathcal{D}(A) \quad t \geq 0$ and

$$A\int_0^t S(t-s)f(s)ds = S(t)f(0) - f(t) + \int_0^t S(t-s)f'(s)ds$$

yielding (2.11) and therefore (ii) is proved.□

Proof of theorem 2.2.2: Let u be defined by (2.10). From Lemma 2.2.1, we conclude that

$$u \in \mathcal{C}([0,\infty);\mathcal{D}(A)) \;\; \textit{and} \;\; Au(t) + f(t) = S(t)(Au_0 + f(0)) + \int_0^t S(t-s)f'(s)ds, \quad t \geq 0$$

Integrating from 0 to t we gets

$$\begin{aligned}
\int_0^t \left(Au(\tau) + f(\tau)\right) d\tau &= \int_0^t S(\tau)(Au_0 + f(0))d\tau + \int_0^t \int_0^\tau S(\tau - s)f'(s)dsd\tau \\
&= \int_0^t \frac{d}{d\tau}\left(S(\tau)u_0\right) d\tau + \int_0^t S(\tau)f(0)d\tau \\
&\quad + \int_0^t S(s)\int_s^t f'(\tau - s)d\tau ds \\
&= S(t)u_0 - u_0 + \int_0^t S(\tau)f(0)d\tau \\
&\quad + \int_0^t S(s)\left(f(t-s) - f(0)\right) ds \\
&= S(t)u_0 - u_0 + \int_0^t S(t-s)f(s)ds = u(t) - u_0, \quad t \geq 0
\end{aligned}$$

That is u is a strong solution of (2.5).

b) Let u be a strong solution of (2.5). For fixed $t>0$ we set $v(s)=S(t-s)u(s),\ 0\leq s\leq t$. For $s\in[0,t)$ and sufficiently small $h>0$ we get

$$\begin{aligned}\frac{1}{h}\big(v(s+h)-v(s)\big) &= \frac{1}{h}\Big(S(t-s-h)u(s+h)-S(t-s)u(s)\Big)\\ &= S(t-(s+h))\frac{1}{h}\big(u(s+h)-u(s)\big)+S(t-s-h)\frac{1}{h}\big(u(s)-S(h)u(s)\big)\end{aligned}$$

As $h\downarrow 0$ this implies

$$\frac{d^+}{ds}v(s)=S(t-s)\big(u'(s)-Au(s)\big)\quad a.e\quad on\quad [0,t)$$

Analogously we get, for $s\in(0,t]$ and $h\in(0,t)$,

$$\frac{d^-}{ds}v(s)=S(t-s)\big(u'(s)-Au(s)\big)\quad a.e\quad on\quad (0,t]$$

so that

$$v'(s)=S(t-s)\big(u'(s)-Au(s)\big)=S(t-s)f(s)\quad a.e\quad on\quad (0,t)$$

Integrating from 0 to t we get (2.10).□

Theorem 2.2.2-b) shows that strong solutions of (2.5) are unique provided A is the infinitesimal generator of a $\mathcal{C}_0$-semigroup and $f\in L^1_{loc}([0,\infty),X)$. Since $\int_0^t S(t-s)f(s)ds$ is well-defined for $f\in L^1_{loc}([0,\infty),X)$, one would be curious to have an idea about the connection between u given by (2.10) and the Cauchy problem (2.5).

For $u_0\in X$ and $f\in L^1_{loc}([0,\infty),X)$ we can find sequences $(u_n^0)_{n\geq1}\subset\mathcal{D}(A)$ (since $\mathcal{D}(A)$ is dense in X) and $(f_n)_{n\geq1}\subset W^{1,1}_{loc}([0,\infty),X)$ with $u_n^0\longrightarrow u_0$ and $\|f-f_n\|_{L^1(0,T;X)}\longrightarrow 0$ for all $T>0$ as $n\longrightarrow\infty$. The problems

$$\begin{cases}\frac{d}{dt}u_n(t) &= Au_n(t)+f_n(t),\quad t>0\\ u_n(0) &= u_n^0\end{cases}$$

have unique strong solutions by Theorem 2.2.2. This motivates the following definition.

Definition 2.2.3 *Let A be a linear operator on X and $f\in L^1_{loc}([0,\infty),X)$. A continuous function $u:[0,\infty)\longrightarrow X$ is called a mild solution of (2.5) if and only if $u(0)=u_0$ and there exist sequences $(f_n)_{n\geq1}\subset L^1_{loc}([0,\infty),X)$, $(u_n)_{n\geq1}\subset\mathcal{C}([0,\infty);X)$ such that u_n is a strong solution of*
$\frac{du_n(t)}{dt}=Au_n(t)+f_n(t)$ and
$\lim_{n\to\infty}\|f-f_n\|_{L^1(0,T;X)}=0$ for all $T>0$,
$\lim_{n\to\infty}u_n(t)=u(t)$ uniformly on $[0,T]$ for any $T>0$.

The following theorem gives a characterization of mild solutions if the operator A is closed:

Theorem 2.2.3 *Assume that A is a closed operator on X and that $f \in L^1_{\text{loc}}([0,\infty),X)$. Then a continous function $u : [0,\infty) \longrightarrow X$ is mild solution of (2.5) if and only if*

$$\int_0^t u(s)ds \in \mathcal{D}(A), \quad t \geq 0 \tag{2.12}$$

and

$$u(t) = u_0 + A\int_0^t u(s)ds + \int_0^t f(s)ds, \quad t \geq 0 \tag{2.13}$$

Proof: a) Assume first that u is a mild solution of (2.5) and choose sequences $(f_n)_{n\geq 1}, \quad (u_n)_{n\geq 1}$ according to Definition 2.2.3. Since u_n are strong solutions, we have $\int_0^t u_n(s)ds \in \mathcal{D}(A), t \geq 0$ and

$$\begin{aligned} A\int_0^t u_n(s)ds + \int_0^t f_n(s)ds &= \int_0^t \big(Au_n(s) + f_n(s)\big)ds \\ &= \int_0^t u_n'(s)ds = u_n(t) - u_n(0), \quad t \geq 0 \end{aligned}$$

which implies
$\lim\limits_{n\to\infty} A\int_0^t u_n(s)ds = u(t) - u_0 - \int_0^t f(s)ds, \quad t \geq 0.$
Obviously we have $\lim\limits_{n\to\infty} \int_0^t u_n(s)ds = \int_0^t u(s)ds$. Closedness of A implies that $\int_0^t u(s)ds \in \mathcal{D}(A)$ and $A\int_0^t u(s)ds = u(t) - u_0 - \int_0^t f(s)ds, \quad t \geq 0$, i.e., u satisfies (2.12) and (2.13).

b) Assume that u satisfies (2.12) and (2.13). For $h > 0$ we define
$u_h(t) = \frac{1}{h}\int_t^{t+h} u(s)ds \quad and \quad f_h(t) = \int_t^{t+h} f(s)ds, \quad t \geq 0.$
The functions u_h are continously differentiable on $[0,\infty)$ with

$$\frac{d}{dt}u_h(t) = \frac{1}{h}\big(u(t+h) - u(t)\big), \quad t \geq 0. \tag{2.14}$$

By standard results we also have, for any $T > 0$.
$\lim\limits_{h\downarrow 0}\|f - f_h\|_{L^1(0,T;X)} = 0.$
From (2.12) we conclude that $u_h \in \mathcal{D}(A), t \geq 0$. Using (2.13) and (2.14) we obtain.

$$\begin{aligned} Au_h(t) &= \frac{1}{h}\Big(A\int_0^{t+h} u(s)ds - A\int_0^t u(s)ds\Big) \\ &= \frac{1}{h}\big(u(t+h) - u(t)\big) - \frac{1}{h}\int_t^{t+h} f(s)ds \\ &= \frac{d}{dt}u_h(t) - f_h(t), \quad t \geq 0, \end{aligned}$$

i.e. u_h is a strong solution $\frac{d}{dt}u = Au + f_h$ on $[0,\infty)$. The estimate

$$\|u(t) - u_h(t)\| \leq \frac{1}{h}\int_t^{t+h} \|u(t) - u(s)\|ds \leq \max_{t\leq s\leq t+h}\|u(t) - u(s)\|$$

proves that $\lim\limits_{h\downarrow 0} u_h(t) = u(t)$ uniformly on intervals [0,T], $T > 0$. Thus we have shown that u is a mild solution of (2.5)$\square$.

Theorem 2.2.4 *Let A be the infinitesimal generator of a $\mathcal{C}_0$-semigroup $S(.)$ and assume that $f \in L^1_{loc}([0,\infty), X)$. Then*

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s)ds, \quad t \geq 0 \tag{2}$$

is the unique mild solution of (2.5).

Proof: Let u be defined by the variation of parameters formula (2) above. Then we have

$$\begin{aligned} \int_0^t u(s)ds &= \int_0^t S(s)u_0 ds + \int_0^t \int_0^s S(s-\tau)f(\tau)d\tau ds \\ &= \int_0^t S(s)u_0 ds + \int_0^t \int_0^{t-\tau} S(\tau)f(s)dsd\tau. \end{aligned}$$

Since A is closedly defined, $\int_0^t S(s)u_0 ds \in \mathcal{D}(A)$, for all $t \in [0,\infty)$ and we obtain by Lemma 2.2.1 that

$$\begin{aligned} A\int_0^t u(s)ds &= S(t)u_0 - u_0 + \int_0^t \big(S(t-\tau)f(\tau) - f(\tau)\big)ds \\ &= u(t) - u_0 - \int_0^t f(\tau)d\tau \end{aligned}$$

which is (2.13).

If u_1, u_2 are two mild solutions of (2.5), then $u = u_1 - u_2$ is a mild solution of the homogeneous problem

$$\frac{d}{dt}u(t) = Au(t), \quad t \geq 0, \quad u(0) = 0$$

According to Definition 2.2.3 of mild solutions there exist a sequence $(f_n) \subset L^1_{\text{loc}}([0,\infty), X)$ with $\|f_n\|_{L^1(0,T;X)} \longrightarrow 0$ for all $T > 0$ and a sequence $(u_n)_{n\geq 1}$ of strong solutions of

$$\frac{d}{dt}u_n = Au_n + f_n \ \text{, with } \lim_{n\to\infty} u_n(t) = u(t)$$

uniformly on intervals $[0,T]$, $T > 0$. Theorem 2.2.2-b), implies that

$$u_n(t) = S(t)u_n(0) + \int_0^t S(t-s)f_n(s)ds, \quad t \geq 0, n = 1, 2, \cdots$$

Since we have that $u(0) = 0$, we get $\lim_{n\to\infty} u_n(0) = 0$, which implies

$$u(t) = \lim_{n\to\infty} u_n(t) = 0, t \geq 0$$

Mild solutions can also be characterized as weak solutions. We recall that for a densely defined linear operator A on X and any element $x^* \in X^*$ (dual of X) there exists at most one element

$$y^* \in X^* \text{ such that } \langle Ax, x^* \rangle = \langle x, y^* \rangle \text{ for all } x \in \mathcal{D}(A).$$

This justifies the following definition of the *adjoint operator* A^* corresponding to a linear operator A with $\overline{\mathcal{D}(A)} = X$

$$\mathcal{D}(A^*) = \Big\{x^* \in X^* | \exists \quad y^* \in X^* \mid \langle Ax, x^*\rangle \; = \; \langle x, y^*\rangle \; \forall x \in \mathcal{D}(A)\Big\},$$

$$A^* x^* = y^*, \qquad x^* \in \mathcal{D}(A)$$

where $y^* \in X^*$ is the unique element with $\langle Ax, x^*\rangle \; = \; \langle x, y^*\rangle$ for all $x \in \mathcal{D}(A)$.

We shall need the following lemma

Lemma 2.2.2 *Let A be a densely defined closed operator on X and $x_0, y_0 \in X$ such that $\langle y_0, x^*\rangle \; = \; \langle x_0, A^*x^*\rangle$ for all $x^* \in \mathcal{D}(A^*)$. Then we have $x_0 \in \mathcal{D}(A)$ and $y_0 = Ax_0$.*

Proof: Let $G_A := \{(x, Ax) \in X \times X | x \in \mathcal{D}(A)\}$ denote the graph of A, and assume that $(x_0, y_0) \notin G_A$. Since G_A is a closed subspace of $X \times X$, the Hahn-Banach theorem assures the existence of two bounded linear functionals $x_1^*, x_2^* \in X^*$ such that

$$\langle x, x_1^*\rangle \; + \; \langle Ax, x_2^*\rangle \; = 0 \quad for \quad all \quad x \in \mathcal{D}(A), \tag{2.15}$$

and

$$\langle x_0, x_1^*\rangle \; + \; \langle y_0, x_2^*\rangle \; \neq \; 0. \tag{2.16}$$

By definition of A^* equation (2.15) implies $x_2^* \in \mathcal{D}(A^*)$ and $A^* x_2^* = -x_1^*$. Using this in (2.16) we get $\langle x_0, A^* x_2^*\rangle \; \neq \; \langle y_0, x_2^*\rangle$, a contradiction to the hypothesis on x_0 and y_0.

Theorem 2.2.5 *Let A be a closed operator on X with dense domain and assume $f \in L^1_{\text{loc}}([0, \infty), X)$. Then a continous function $u : [0, \infty) \longrightarrow X$ is a mild solution of (2.5) if and only if u is a weak solution of (2.5),i.e. u is continuous on $[0, \infty)$ with the following properties:*

(i) $u(0) = u_0$.

(ii) *For all $x^* \in \mathcal{D}(A^*)$ the function $t \mapsto \langle u(t), x^*\rangle$ is absolutely continous on all intervals [0,T], $T > 0$, and*

$$\frac{d}{dt} \langle u(t), x^*\rangle \; = \; \langle u(t), A^* x^*\rangle \; + \; \langle f(t), x^*\rangle \quad a.e \quad on \quad [0, \infty) \tag{2.17}$$

Proof: a) Let u be a mild solution of (2.5). For any $x^* \in \mathcal{D}(A^*)$ we get from (2.12) $\Big\langle A\int_0^t u(s)ds, x^*\Big\rangle = \Big\langle \int_0^t u(s)ds, A^*x^*\Big\rangle, \quad t \geq 0$. Besides by (2.13) we have for all $x^* \in \mathcal{D}(A^*)$,

$$\begin{aligned}
\langle u(t), x^*\rangle \;&=\; \langle u_0, x^*\rangle + \Big\langle A\int_0^t u(s)ds, x^*\Big\rangle + \Big\langle \int_0^t f(s)ds, x^*\Big\rangle \\
&=\; \langle u_0, x^*\rangle + \Big\langle \int_0^t u(s)ds, A^*x^*\Big\rangle + \Big\langle \int_0^t f(s)ds, x^*\Big\rangle \\
&=\; \langle u_0, x^*\rangle + \int_0^t \big(\langle u(s), A^*x^*\rangle \; + \; \langle f(s), x^*\rangle\big)ds, \quad t \geq 0
\end{aligned}$$

This implies that $t \mapsto \langle u(t), x^* \rangle$ is absolutely continous on interval $[0,T]$, $T > 0$, and (2.17) holds.
b) If u is a weak solution, then integrating (2.17) from 0 to t we get

$$\left\langle u(t) - u_0 - \int_0^t f(s)ds, x^* \right\rangle = \left\langle \int_0^t u(s)ds, A^*x^* \right\rangle \text{ for all t} \geq 0 \text{ and x}^* \in \mathcal{D}(\mathrm{A}^*)$$

According to Lemma 2.2.2 we have

$\int_0^t u(s)ds \in \mathcal{D}(A)$ and $A\int_0^t u(s)ds = u(t) - u_0 - \int_0^t f(s)ds$.

The following theorem shows that the existence and uniqueness of a mild solution is also a sufficient condition for a closed linear operator to be the infinitesimal generator of $\mathcal{C}_0$-semigroup.

Theorem 2.2.6 (Ball) *Let A be a closed linear operator on X and assume that $f \in L^1_{\text{loc}}([0,\infty); X)$. The operator A is the infinitesimal generator of a $\mathcal{C}_0$-semigroup. $S(\cdot)$ if and only if the abstract Cauchy problem (2.5) has for any $u_0 \in X$ a unique mild solution $u(,;u_0)$. In this case* $u(t;u_0) = S(t)u_0 + \int_0^t S(t-s)f(s)ds, \quad t \geq 0$

Proof: In view of Theorem 2.2.4 we have only to prove that existence and uniqueness of mild solutions imply that A is the infinitesimal generator. The assumptions imply that the homogeneous problem

$$\begin{cases} \frac{d}{dt}v(t) &= Av(t), \quad t > 0 \\ v(0) &= x_0 \end{cases} \tag{2.18}$$

has a unique mild solution for any $x_0 \in X$, which we denote by $v(t;x_0)$. We define the operators $S(t)$, $t \geq 0$ by $S(t)x_0 = v(t;x_0)$, $\quad t \geq 0$. It is easily seen that $S(0) = I$. In order to prove the semigroup property we choose $x_0 \in X$ and $t \geq s \geq 0$ and define

$$u_1(t) = v(t+s;x_0) = S(t+s)x_0 \textit{ and } u_2(t) = v(t;v(s;x_0)) = S(t)S(s)x_0$$

Obviously u_2 is the unique mild solution of (2.18) with initial value $v(0) = v(s;x_0)$. For u_1, we have $u_1(0) = v(s;x_0)$ and using (2.12)

$$\begin{aligned} \int_0^t u_1(\tau)d\tau &= \int_0^t v(\tau+s;x_0)d\tau \\ &= \int_s^{s+t} v(\tau;x_0)d\tau \text{ (by a change of variable)} \\ &= \int_0^{s+t} v(\tau;x_0)d\tau - \int_0^s v(\tau;x_0)d\tau \in \mathcal{D}(A) \end{aligned}$$

Using (2.13) and taking $u_0 = x_0$ we get also

$$\begin{aligned} u_1(t) = v(t,s,x_0) &= x_0 + A\int_0^{t+s} v(\tau;x_0)d\tau \\ &= x_0 + A\int_0^s v(\tau;x_0)d\tau + A\int_0^t v(\tau+s;x_0)d\tau \\ &= v(s;x_0) + A\int_0^t u_1(\tau)d\tau, \qquad t \geq 0 \end{aligned}$$

Theorem 2.2.3 implies that u_1 is a mild solution of (2.18) with initial value $v(s;x_0)$. By uniqueness of mild solutions we have $u_1 = u_2$, i.e., $S(t+s) = S(t)S(s)$.

Obviously, the function $t \mapsto S(t)x_0$ is continuous on $[0,\infty)$ for any $x_0 \in X$. In order to prove that the operators $S(t)$ are bounded,
we define for any $T > 0$ the mapping $\Phi_T : X \to \mathcal{C}(0,T;X)$ by

$$(\Phi_T x)(t) = v(t;x) = S(t)x, \quad t \in [0,T], \quad x \in X$$

Let $(x_n)_{n\geq 1}$ be a sequence in X with $x_n \to x$ and $\Phi_T x_n \to w \in \mathcal{C}(0,T;X)$ as $n \to \infty$. For any n there exists a unique mild solution $v(.;x_n)$ of (2.18) with initial value x_n. By theorem 2.2.3 we have

$$\int_0^t (\Phi_T x_n)(s)ds \in \mathcal{D}(A) \ \ and \ \ A\int_0^t (\Phi_T x_n)(s)ds = (\Phi_T x_n)(t) - x_n, \quad t \geq 0, n = 1,2,3,\cdots$$

which implies $\lim_{n\to\infty} A\int_0^t (\Phi_T x_n)(s)ds = w(t) - x, \quad 0 \leq t \leq T$. It is obvious by Dominated Convergence Theorem that

$$\lim_{n\to\infty} \int_0^t (\Phi_T x_n)(s)ds = \int_0^t w(s)ds, \quad 0 \leq t \leq T$$

Closedness of A implies that

$$\int_0^t w(s)ds \in \mathcal{D}(A) \ \ and \ \ A\int_0^t w(s)ds = w(t) - x, \quad 0 \leq t \leq T.$$

According to Theorem 2.2.3, w is the restriction of the unique mild solution of (2.18) with initial value x to $[0,T]$, i.e. $w = \Phi_T x$. Therefore Φ_T is a closed operator defined on all of X. By the closed graph theorem, Φ_T is bounded and consequently $\|S(\cdot)\|$ is bounded on any interval $[0,T]$. Thus $S(\cdot)$ is a $\mathcal{C}_0$-semigroup on X.

Let B denote the infinitesimal generator of $S(\cdot)$. We have to prove that $A = B$. We first choose $x \in \mathcal{D}(B)$. Then we have

$$\lim_{h\downarrow 0} \frac{1}{h}(S(h)x - x) = Bx \tag{2.19}$$

On the other hand $v(\cdot) = S(\cdot)x$ is the unique mild solution of (2.18) with $v(0) = x$. By Theorem 2.2.3 this implies

$$\frac{1}{h}(S(h)x - x) = A\frac{1}{h}\int_0^h S(s)xds, \quad h > 0$$

This together with (2.19) implies $\lim_{h\to 0} A\Big(\frac{1}{h}\int_0^h S(s)xds\Big) = Bx$. Furthermore, we have $\lim\limits_{h\downarrow 0}\frac{1}{h}\int_0^h S(s)xds = x$. Thus closedness of A implies $x \in \mathcal{D}(A)$ and $Ax = Bx$, i.e. A is an extension on B.

Let now $x \in \mathcal{D}(A)$. Then $S(t)x$ and $S(t)Ax$ are mild solution of (2.18) and consequently, by Theorem 2.2.3,

$$A\int_0^t S(s)xds = S(t)x - x \quad and \quad A\int_0^t S(s)Axds = S(t)Ax - Ax, \forall t \geq 0 \tag{2.20}$$

Since $s \mapsto \int_0^s S(\tau)Axd\tau$ is a continuous mapping into $\mathcal{D}(A)$ and A is closed, we have

$$A\int_0^t \int_0^s S(\tau)Axd\tau ds = \int_0^t A\int_0^s S(\tau)Axd\tau ds = \int_0^t \big(S(s)Ax - Ax\big)ds, t \geq 0 \tag{2.21}$$

We define

$$z(t) := \int_0^t S(s)Axds - A\int_0^t S(s)xds = \int_0^t S(s)Axds - S(t)x + x, \quad t > 0$$

Obviously, z is continuous on $t \geq 0$ with $z(0) = 0$. Using (2.20) and (2.21) we get

$$\begin{aligned} A\int_0^t z(s)ds &= A\int_0^t \int_0^s S(\tau)Axd\tau ds - A\int_0^t A\int_0^s S(\tau)xd\tau ds \\ &= \int_0^t \big(S(s)Ax - Ax\big)ds - A\int_0^t \big(S(s)x - x\big)ds \\ &= \int_0^t S(s)Axds - A\int_0^t S(s)xds = z(t), \quad t \geq 0 \end{aligned}$$

According to Theorem 2.2.3 this proves that z is a generalized solution of (2.18) with initial value $z(0) = 0$. By uniqueness of generalized solutions we have $z(t) \equiv 0$ on $t \geq 0$ and consequently

$$\frac{1}{t}\big(S(t)x - x\big) = \frac{1}{t}\int_0^t S(s)Axds, \quad t \geq 0.$$

For $t \downarrow 0$ we get $x \in \mathcal{D}(B)$ and the proof is complete.

In this chapter we studied linear Evolution Equations when the forcing term is say easier to handle. We now move to the next chapter where we concentrate more on the case in which the forcing term is state dependent.

CHAPTER 3

SEMI-LINEAR EVOLUTION EQUATIONS

3.1 Introduction

In this chapter we study another class of evolution equations in which the forcing term depends on the state of the system at some time t. We consider the following Cauchy problem:

$$\begin{cases} u'(t) &= Au(t) + f(t, u(t)), \quad t > 0 \\ u(0) &= u_0 \end{cases} \tag{3.1}$$

where A is the infinitesimal generator of a $\mathcal{C}_0$-semigroup denoted by $\{e^{tA}, t \geq 0\}$ and $f : [0, T] \times \mathbf{X} \to \mathbf{X}$ is continuous.

In the linear case we need the forcing term to just be continuous to guarantee the existence of a mild solution. But in this present case, we will require more than continuity on f to have existence of a solution, as we can see in the following example.

Example: In (3.1) above, let $A = 0$ and $X = \mathcal{C}_0$ the Banach space of all real-valued sequences $u = \{\xi_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} \xi_n = 0$ and $\|u\| = \sup_{n\geq 1} |\xi_n|$. Define the function

$$f : X \to X \; by \; f(u) = \{|\xi_n|^{\frac{1}{2}} + n^{-1}\}_{n=1}^{\infty}, \quad u = \{\xi_n\}_{n=1}^{\infty} \in X.$$

The continuity of the function $\xi \mapsto \xi^{\frac{1}{2}}$ for $\xi \geq 0$ and the definition of the norm on X imply that f is continous on X. But the initial value problem

$$\frac{du}{dt} = f(u), \quad u(0) = 0 \tag{3.2}$$

has no solution in X.
Suppose on the contrary that $u = \{\xi_n\}_{n=1}^{\infty}$ is a solution of (3.2). Then the n^{th} coordinate would satisfy the scalar equation

$$\xi_n'(t) = |\xi_n|^{\frac{1}{2}} + n^{-1} \tag{3.3}$$

and the initial condition

$$\xi_n(0) = 0 \tag{3.4}$$

From (3.3), $\xi_n'(t) \geq 0$ and so $\xi_n(t)$ is strictly increasing in t and in view of (3.4), $\xi_n(t) > 0$ for $0 < t < \tau$ where τ is sufficiently small. Then it follows from (3.2) that

$$\xi_n'(t) > \xi_n^{\frac{1}{2}}(t), \quad 0 < t < \tau$$

which leads to $\xi_n(t) \geq \frac{1}{4}t^2, 0 < t < \tau$. And one easily sees that no matter how small we choose τ, the sequence $\{\xi_n\}_{n=1}^{\infty}$ does not converge to zero as $n \to \infty$ which contradicts the hypothesis that u is a solution of (3.2), and in particular $u(t) \in X$. Thus although the function f is continous, the initial value problem (3.2) has no solution in any interval of the form $[0, T]$.

We will see that when f satisfies a Lipschitz-type condition, we have existence and uniqueness of a mild solution, which under appropriate conditions can become a classical solution and that when f does not satisfy any Lipschitz-type condition, more properties are required from the linear operator A to guarantee merely the existence of a mild solution.
By identification to the Variation of parameter formula, a mild solution to (3.1) satisfies the equation:

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A} f(s, u(s))ds. \tag{3.5}$$

Solving (3.1) boils down to solving equation (3.5) and the self-referential nature of (3.1) makes the situation more complicated as we need to examine the behavior of the mapping $t \longmapsto \int_0^t e^{(t-s)A} f(s, u(s))ds$.

3.2 Theory for Lipschitz-Type Forcing terms

3.2.1 Existence and uniqueness results

We now state and prove one of the main results of this section. Consider the following hypothesis.

(H_A) $A : \mathcal{D}(A) \subset \mathbf{X} \to \mathbf{X}$ ***generates a*** $\mathcal{C}_0$***-semigroup denoted by*** $\{e^{tA} : t \geq 0\}$ ***on*** $\mathbf{X}$ ***and assume*** $\overline{M_A} = \sup_{t\geq 0}\|e^{tA}\| < \infty$.

Theorem 3.2.1 (Existence and Uniqueness) [1] *Suppose hypothesis* (H_A) *holds and assume,* (H_1), $f : [0, T] \times \mathbf{X} \to \mathbf{X}$ *is continuously Fréchet differentiable on the product space* $[0, T] \times \mathbf{X}$

i) *If* $u_0 \in \mathbf{X}$ *and* f *is globally Lipschitz with respect to the second variable (uniformly in* t*),then (3.1) has a unique mild solution* u *on* $[0, T]$ *which satisfies (3.5).*

ii) *If* $u_0 \in \mathcal{D}(A)$ *and* (H_1) *holds, then the mild solution in (i) is also a classical solution of (3.1).*

Proof:
i) Consider the solution map ϕ defined by

$$\phi : \mathcal{C}([0, T], \mathbf{X}) \to \mathcal{C}([0, T], \mathbf{X}), \quad (\phi u)(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A} f(s, u(s))ds$$

Given $u \in \mathcal{C}([0,T],\mathbf{X})$, observe that $\phi(u)$ is continuous as the sum of two continuous functions. We shall show that ϕ has a unique fixed point. Let $u \in \mathcal{C}([0,T],\mathbf{X})$ and note that

$$\begin{aligned}
\sup_{0\le t\le T}\|(\phi u)(t)\| &\le \sup_{0\le t\le T}\left[\|e^{tA}u_0\| + \left\|\int_0^t e^{(t-s)A}f(s,u(s))ds\right\|\right]\\
&\le Me^{\alpha T}\Big(\|u_0\| + \int_0^t \|f(s,u(s))\|ds\Big)\\
&\le Me^{\alpha T}\Big(\|u_0\| + \int_0^t \|f(s,u(s)) - f(s,u_0) + f(s,u_0)\|ds\Big)\\
&\le Me^{\alpha T}\Big(\|u_0\| + \int_0^t \big[M_f\|u(s)-u_0\| + \|f(s,u_0)\|\big]ds\Big)\\
&< \infty
\end{aligned}$$

Now let $u, v \in \mathcal{C}([0,T],\mathbf{X})$.
We have that $\forall t \in [0,T]$.

$$\begin{aligned}
\|(\phi u)(t) - (\phi v)(t)\| &\le \int_0^t \|e^{(t-s)A}\|\|f(s,u(s)) - f(s,v(s))\|ds\\
&\le \int_0^t C\|e^{(t-s)A}\|\|u(s)-v(s)\|ds\\
&\le \int_0^t C\|e^{(t-s)A}\| \sup_{0\le s\le T}\|u(s)-v(s)\|ds\\
&\le CM_0\|u-v\|\int_0^t \|e^{w(t-s)}\|ds\\
&\le CM_0\|u-v\|\frac{1}{w}\left(e^{wt}-1\right)\\
&\le CM_0\|u-v\|\frac{1}{w}e^{wt}.
\end{aligned}$$

Taking the supremum over $[0,T]$, we have that:

$$\|\phi(u)-\phi(v)\| \le CM_0\frac{1}{w}e^{wT}\|u-v\|$$

So for $CM_0\frac{1}{w}e^{wT} < 1$, ϕ is a contraction and therefore has a unique fixed-point. Hence choosing τ such that $CM_0\frac{1}{w}e^{w\tau} < 1$, (3.1) has a unique mild solution on $[0,T]$.

ii) Now assume that $u_0 \in \mathcal{D}(A)$ and let u be the mild solution of (3.1) guaranteed to exist by (i). We argue that u given by (3.5) is differentiable, satisfies (3.1) and is such that

$$u(t) \in \mathcal{D}(A), \forall t \in [0,T]$$

Consider the following auxiliary I.V.P (Variational problem).

$$\begin{cases} z'(t) + Az(t) = [\frac{\partial}{\partial u}f(t,u(t))]z(t), & 0 < t < T\\ z(0) = f(0,u_0) - Au_0 \end{cases} \tag{3.6}$$

By hypothesis, $\frac{\partial f}{\partial u}(\cdot, u(\cdot))$ is continuous and $u \in \mathcal{C}([0,T],\mathbf{X})$. From part (i) since f is globally Lipschitz with respect to the second variable, the mapping $z \mapsto [\frac{\partial f}{\partial u}(\cdot, u(\cdot))]z$ is also globally Lipschitz on $\mathcal{C}([0,T],\mathbf{X})$. Let $g(t,y) = \frac{\partial}{\partial u}f(t,u(t))y$.

Hence by part (i) of the theorem, (3.6) has a unique mild solution $z \in \mathcal{C}([0,T],\mathbf{X})$ given by

$$z(t) \;=\; e^{tA}(f(0,u_0) - Au_0) + \int_0^t e^{(t-s)A}\Big[\frac{\partial f}{\partial u}(s,u(s))\Big]z(s)ds, \qquad 0 \le t \le T \tag{3.7}$$

We argue that $z(t) = u'(t),\ \forall t \in [0,T]$.

That is

$$\lim_{h\to 0}\Big\|\frac{u(t+h)-u(t)}{h} - z(t)\Big\| = 0, \quad \forall t \in (0,T]. \tag{3.8}$$

Let $\epsilon > 0$ *and* $t \in [0,T]$ using (3.1) in (3.8), we get:

$$\begin{aligned}
\Big\|\frac{u(t+h)-u(t)}{h} - z(t)\Big\| \quad &= \quad \Big\|\frac{e^{(t+h)A}u_0}{h} + \frac{1}{h}\int_0^{t+h} e^{(t+h-s)A}f(s,u(s))ds \\
&- \quad \frac{e^{tA}u_0}{h} - \frac{1}{h}\int_0^t e^{(t-s)A}f(s,u(s))ds \\
&- \quad e^{tA}(f(0,u_0) - Au_0) - \int_0^t e^{(t-s)A}\left[\frac{\partial}{\partial u}f(s,u(s))\right]z(s)ds\Big\| \\
&= \quad \Big\|(\frac{e^{(t+h)A} - e^{tA}}{h})u_0 - Au_0e^{tA} + \frac{1}{h}\int_0^h e^{(t+h-s)A}f(s,u(s))ds \\
&- \quad f(0,u_0)e^{tA} + \frac{1}{h}\int_0^{t+h} e^{(t+h-s)A}f(s,u(s))ds \\
&- \quad \frac{1}{h}\int_0^h e^{(t-s)A}f(s,u(s))ds - \int_0^t e^{(t-s)A}\left[\frac{\partial}{\partial u}f(s,u(s))\right]z(s)ds\Big\| \\
&= \quad \Big\|J_1(h) + J_2(h) + J_3(h)\Big\|
\end{aligned}$$

Where $J_1(h) = (\frac{e^{(t+h)A} - e^{tA}}{h})u_0 - Au_0e^{tA}$

$J_2(h) = \frac{1}{h}\int_0^h e^{(t+h-s)A}f(s,u(s))ds - f(0,u_0)e^{tA}$

$J_3(h) = \frac{1}{h}\int_0^{t+h} e^{(t+h-s)A}f(s,u(s))ds - \frac{1}{h}\int_0^h e^{(t-s)A}f(s,u(s))ds - \int_0^t e^{(t-s)A}[\frac{\partial}{\partial u}f(s,u(s))]\cdot z(s)ds$
We shall apply Gronwall' lemma to an inequality of the form
$\|k(t)\| \le M(\epsilon) + C\int_0^t \|k(s)\|ds$ where $M(\epsilon)$ depends only on ϵ, C is a positive constant and

$$k(t) = \frac{u(t+h)-u(t)}{h} - z(t)$$

Applying this lemma requires us to establish several estimates involving $\|J_i(h)\|_{\mathbf{X}}$ for $i = 1,2,3$. We first note that since $u, \frac{\partial f}{\partial u} \in \mathcal{C}([0,T],\mathbf{X}), \exists C > 0$ such that

$$\sup_{0\le s\le T}\Big\|\frac{\partial f}{\partial u}(s,u(s))\Big\| \le C \tag{3.9}$$

Since we have that:

$$\begin{aligned}\Big\|\frac{\partial f}{\partial u}(s,u(s))\Big\| &= \lim_{h\to 0}\Big\|\frac{f(s,u(s)+h)-f(s,u(s))}{h}\Big\| \\ &\le \lim_{h\to 0} C\Big\|\frac{u(s)+h-u(s)}{h}\Big\| \\ &= C\end{aligned}$$

Also $\exists \delta_1, \delta_2 > 0$ for which we have

$$0 <\mid h \mid< \delta_1 \Rightarrow \|J_1(h)\|_{\mathbf{X}} < \frac{\epsilon}{3e^{k_0\overline{M}_A T}} \tag{3.10}$$

$$0 <\mid h \mid< \delta_2 \Rightarrow \|J_2(h)\|_{\mathbf{X}} < \frac{\epsilon}{3e^{k_0\overline{M}_A T}} \tag{3.11}$$

Now we estimate also $J_3(h)$ in the following way:
Adding and substracting the term $\int_0^t e^{(t-s)A}[\frac{\partial f}{\partial u}(\cdot, u(\cdot))](s)[\frac{u(t+h)-u(t)}{h}]ds$ and performing the change of variable $w = s - h$ in the first three of the integrals in the expression of $J_3(h)$, we obtain.

$$\begin{aligned}J_3(h) &= \frac{1}{h}\int_0^t e^{(t-w)A} f(w+h, u(w+h))dw - \frac{1}{h}\int_0^t e^{(t-s)A} f(s,u(s))ds \\ &- \int_0^t e^{(t-s)A}\Big[\frac{\partial f}{\partial u}(s,u(s))\Big]\Big[\frac{u(s+h)-u(s)}{h}\Big]ds \\ &+ \int_0^t e^{(t-s)A}\Big[\frac{\partial f}{\partial u}(s,u(s))\Big]\Big[\frac{u(s+h)-u(s)}{h} - z(s)\Big]ds\end{aligned}$$

Since w is a dummy variable, we can still label it s and combine the first three integrals on the Right Hand Side of the above equality. Doing so and applying the triangle inequality yields.
$\|J_3(h)\| \le \|J_4(h)\| + \|J_5(h)\|$ where

$J_4(h) = \int_0^t e^{(t-s)A}\frac{1}{h}\Big\{f(s+h, u(s+h))ds - f(s,u(s)) - \Big[\frac{\partial f}{\partial u}(s,u(s))\Big]\Big[\frac{u(s+h)-u(s)}{h}\Big]\Big\}ds$

$J_5(h) = \int_0^t e^{(t-s)A}\Big[\frac{\partial f}{\partial u}(s,u(s))\Big]\Big[\frac{u(s+h)-u(s)}{h} - z(s)\Big]ds$
Making use of the triangle inequality and the regularity of u and $\frac{\partial f}{\partial u}$, we can show that $\exists \delta_3 > 0$ such that

$0 <\mid h \mid< \delta_3 \Rightarrow \|J_4(h)\|_{\mathbf{X}} < \frac{\epsilon}{3e^{k_0\overline{M}_A T}}$ furthermore

$\|J_5(h)\|_{\mathbf{X}} \le k_0\overline{M}_A \int_0^t \|\frac{u(s+h)-u(s)}{h} - z(s)\|ds$

Let $\delta = min\{\delta_1, \delta_2, \delta_3\} > 0$, then $\forall 0 <\mid h \mid< \delta$

$\Big\|\frac{u(t+h)-u(t)}{h} - z(t)\Big\| \le \frac{\epsilon}{e^{k_0\overline{M}_A T}} + k_0\overline{M}_A \int_0^t \Big\|\frac{u(s+h)-u(s)}{h} - z(s)\Big\|ds$
Hence applying Gronwall's Lemma in the inequality above we get

$\Big\|\frac{u(t+h)-u(t)}{h} - z(t)\Big\| \le \frac{\epsilon}{e^{k_0\overline{M}_A T}} e^{\int_0^t k_0\overline{M}_A} < \epsilon, \forall t \in [0,T]$. Therefore we conclude that u is continuously differentiable and thus the I.V.P :

$$\begin{cases} y'(t) + Ay(t) = f(t,u(t)), & 0 < t < T \\ y(0) = u_0 \end{cases} \tag{3.12}$$

has a unique classical solution satisfying

$$y(t) = e^{t\mathbf{A}}u_0 + \int_0^t e^{(t-s)\mathbf{A}}f(s,u(s))ds. \tag{3.13}$$

Since $u(.)$ is unique and is expressed by the Right Hand Side of both (3.5) and (3.13). These two expressions must be equal, thereby showing that $u = y$. As such we conclude that u is the unique classical solution of (3.1). This completes the proof of the Theorem.

Now consider the following initial value problem:

$$\begin{cases} \frac{du}{dt} + Au &= F(u), \qquad 0 < t < T \\ u(0) &= u_0 \end{cases} \tag{3.14}$$

where A is a maximal accretive operator from a dense subset $\mathcal{D}(A)$ in a Banach space X into X, and F is a nonlinear operator from X into X. We have the following theorem which we shall apply later

Theorem 3.2.2 *Suppose that F satisfies the global Lipschitz condition, i.e., there is a positive constant L such that for all $u, v \in X$,*

$$\|F(u) - F(v)\| \leq L\|u - v\|. \tag{3.15}$$

Furthermore, suppose that $u_0 \in X$. Then problem (3.14) admits a unique global mild solution u such that u belongs to $\mathcal{C}([0,+\infty), X)$ and satisfies the following integral equation:

$$u(t) = S(t)u_0 + \int_0^t S(t-\tau)F(u(\tau))d\tau. \tag{3.16}$$

Moreover, given $u_0\, \hat{u}_0 \in X$, the cooresponding respective mild solutions u and $\hat{u}_0$ satisfy

$$\|u(t) - \hat{u}(t)\| \leq e^{Lt}\|u_0 - \hat{u}_0\| \, \forall t \geq 0 \tag{3.17}$$

Also if F is of class $\mathcal{C}^1$, then the mild solution becomes a classical solution.

Proof: We use the contraction mapping theorem as in the proof of the previous theorem. Let

$$\phi(v) = S(t)u_0 + \int_0^t S(t-\tau)F(v(\tau))d\tau \tag{3.18}$$

and

$$\mathcal{E} = \{v \in \mathcal{C}([0,+\infty), X) | \quad \sup_{t\geq 0}\|u(t)\|e^{-kt} < \infty\} \tag{3.19}$$

where k is a positive cocnstant greater than L. In $\mathcal{E}$, we introduce the following norm:

$$\|u\|_{\mathcal{E}} = \sup_{t\geq 0} e^{-kt}\|u(t)\| \tag{3.20}$$

Clearly $\mathcal{E}$ is a Banach space and one easily shows that ϕ defined by (3.18) maps $\mathcal{E}$ into itself, and is a contraction. And therefore by the contraction mapping theorem (3.16) has a unique mild solution in $\mathcal{E}$. Uniqueness also follows by Gronwall's lemma as well as the estimates (3.17)

3.2.2 Existence of Local Solution

In this section we study the existence of local solution. In the previous theorem, we have got an existence and uniqueness result for a global mild solution.
The global Lipschitzness of f gives an idea on how to get a T for the interval [0,T] in the theorem. In this section we restrict the growth condition on the forcing term f, to a local Lipschitz condition.

Proposition 3.2.1 *Assume condition* (H_A) *and*
(H_2) $f : [0,T]\times\mathbf{X} \to \mathbf{X}$ *is continuous in the first variable and locally Lipschitz in the second variable (uniformly in* t*).*
Then there exists a unique local mild solution of (3.1)

Proof
Let $R > 0$ and $t_0 \in (0,T]$ such that ϕ is a contraction. Then by the local Lipschitz preoperty of f, we have that $\forall x_0 \in \mathbf{X}, \exists k_0 > 0$ such that

$$\|f(t,x) - f(t,y)\|_{\mathbf{X}} \leq k_0\|x-y\|_{\mathbf{X}}, \forall x,y \in \overline{\mathcal{B}_{\mathbf{X}}}(x_0,R), \ \ t\in[0,T]$$

Choose $Y = \mathcal{C}([0,t_0]; \overline{\mathcal{B}_{\mathbf{X}}}(x_0,R))$ equiped with the supremum norm.

Define $\phi : Y \subset \mathcal{C}([0,T];\mathbf{X}) \to \mathcal{C}([0,T];\mathbf{X})$, *by* $\phi(u) = e^{tA}u_0 + \int_0^t e^{(t-s)A}f(s,u(s))ds.$
Let $u \in Y$. Then $\phi(u)$ is continuous as the sum of two continuous functions and moreover,

$$\begin{aligned}
\sup_{0\leq t\leq t_0}\|(\phi u)(t)\| &\leq \sup_{0\leq t\leq t_0}\left[\|e^{tA}u_0\| + \left\|\int_0^t e^{(t-s)A}f(s,u(s))ds\right\|\right] \\
&\leq Me^{\alpha t_0}\left(\|u_0\| + \int_0^t \|f(s,u(s))\|ds\right) \\
&\leq Me^{\alpha t_0}\left(\|u_0\| + \int_0^t \|f(s,u(s)) - f(s,u_0) + f(s,u_0)\|ds\right) \\
&\leq Me^{\alpha t_0}\left(\|u_0\| + \int_0^t \left[k_0\|u(s)-u_0\| + \|f(s,u_0)\|\right]ds\right) \\
&\leq Me^{\alpha t_0}\left(\|u_0\| + \int_0^t [k_0R + \|f(s,u_0)\|]ds\right) \\
&< +\infty
\end{aligned}$$

Hence ϕ is well-defined and is continuous as the sum of two continuous functions. Moreover, we have that

$$\begin{aligned}
\|(\phi u)(t) - u_0\|_{\mathbf{Y}} &\leq \|e^{tA}u_0 - u_0\| + \int_0^t \|e^{(t-s)A} f(s, u(s))\| ds \\
&\leq \|e^{tA}u_0 - u_0\| + \overline{M_A} \int_0^t \|f(s, u(s)) - f(s, u_0) + f(s, u_0)\| ds \\
&\leq \|e^{tA}u_0 - u_0\| + \overline{M_A} \int_0^t \|f(s, u(s)) - f(s, u_0) + f(s, u_0)\| ds + \overline{M_A} \int_0^t \|f(s, u_0)\| ds \\
&\leq \|e^{tA}u_0 - u_0\| + \overline{M_A}\Big[K_0 R t + \int_0^t \|f(s, u_0)\| ds\Big]
\end{aligned}$$

Taking the supremum over $[0, t_0]$ we have

$$\begin{aligned}
\|\phi(u) - u_0\|_\infty &\leq \|e^{t_0 A}u_0 - u_0\| + \overline{M_A} \int_0^{t_0} \|f(s, u_0)\| ds + \overline{M_A} K_0 t_0 R \\
&= \alpha_1(t_0) + \alpha_2(t_0) R \qquad (1)
\end{aligned}$$

where $\alpha_1(t_0) = \|e^{t_0 A}u_0 - u_0\| + \overline{M_A} \int_0^{t_0} \|f(s, u_0)\| ds$ and $\alpha_2(t_0) = \overline{M_A} K_0 t_0$
Also let $x, y \in \mathbf{Y}$. Then we have that.

$$\begin{aligned}
\|(\phi x)(t) - (\phi y)(t)\|_{\mathbf{Y}} &\leq \int_0^t \|e^{(t-s)A}\| \|f(s, x(s)) - f(s, y(s))\| ds \\
&\leq \overline{M_A} K_0 \int_0^t \|x(s) - y(s)\| ds \\
&\leq \overline{M_A} K_0 \int_0^t \sup_{0 \leq s \leq T} \|x(s) - y(s)\| ds \\
&\leq \overline{M_A} K_0 \int_0^t \|x - y\| ds \\
&\leq \overline{M_A} K_0 \|x - y\| t
\end{aligned}$$

Taking the supremum over $[0,t_0]$, we get
$\|\phi x - \phi y\|_{\mathbf{Y}} \leq \overline{M_A} K_0 \|x - y\| = \alpha_3(t_0)\|x - y\|_{\mathbf{Y}}$ (2)
where $\alpha_3(t_0) = \overline{M_A} K_0$.
Observe that from (1) and (2) the functions $\alpha_i : [0, T] \to \mathbb{R}^+, i = 1, 2, 3.$ are non negative and $\lim_{t_0 \to 0^+} \alpha_i(t_0) = 0$.
So it follows that

$x \in \mathbf{Y} \Rightarrow \|\phi(x) - u_0\|_{\mathbf{Y}} \leq \alpha_1(t_0) + \alpha_2(t_0) R \leq R$ (3)

$x, y \in \mathbf{Y} \Rightarrow \|\phi x - \phi y\|_{\mathbf{Y}} \leq \alpha_3(t_0)\|x - y\|_{\mathbf{Y}}$ (4)
From (3) we have that

$\forall x \in \mathbf{Y}, \phi(x) \in \mathbf{Y} \Rightarrow \phi(\mathbf{Y}) \subset \mathbf{Y}$. So we can restrict the domain of ϕ and have $\phi : \mathbf{Y} \to \mathbf{Y}$
Now choosing $t_0 \in (0, T]$ such that (3) holds and $\alpha_3(t_0) < 1$ yields that ϕ is a contraction and therefore has a unique fixed-point u, which is a mild solution of (3.1) on $[0, t_0]$. It is called the local mild solution and this completes the proof of the theorem.

3.2.3 Continuous Dependence

Consider (3.1) together with the following IVP

$$\begin{cases} v'(t) = Av(t) + \hat{f}(t, v(t)), \quad 0 < t < T \\ v(0) = v_0 \end{cases} \tag{3.21}$$

Where $\hat{f} : [0, T] \times \mathbf{X} \to \mathbf{X}$ *and* $v_0 \in \mathbf{X}$. We have the following continuous dependence result.

Proposition 3.2.2 *Assume* (H_A) *holds and*
(H_3): *f and $\hat{f}$ are glogally Lipschitz on $\mathbf{X}$ (uniformly in t).*
(H_4): *There exists $\epsilon_1 > 0$ such that* $\sup\limits_{0 \le t \le T} \|f(t,x) - \hat{f}(t,y)\|_{\mathbf{X}} < \epsilon_1, \forall x \in \mathbf{X}$.
(H_5): *There exists $\epsilon_2 > 0$ such that* $\|u_0 - v_0\| < \epsilon_2$.
Then $\|u(t) - v(t)\|_{\mathbf{X}} \le \overline{M_A}(\epsilon_1 T + \epsilon_2) e^{\overline{M_A} K t}, \ 0 \le t \le T$.

Proof: Consider the variation of parameter formulae for (3.1) and (3.21)

$u(t) = e^{tA} u_0 + \int_0^t e^{(t-s)A} f(s, u(s)) ds$

$v(t) = e^{tA} v_0 + \int_0^t e^{(t-s)A} \hat{f}(s, v(s)) ds$

We have that.

$$\begin{aligned}
\|u(t) - v(t)\|_{\mathbf{X}} &\le \|e^{tA}\| \|u_0 - v_0\|_{\mathbf{X}} + \int_0^t \|e^{(t-s)A}\| \|f(s, u(s)) - \hat{f}(s, v(s))\| ds \\
&\le \overline{M_A} \|u_0 - v_0\|_{\mathbf{X}} + \int_0^t \overline{M_A} \|f(s, u(s)) - \hat{f}(s, u(s)) + \hat{f}(s, u(s)) + \hat{f}(s, v(s))\| ds \\
&\le \overline{M_A} \|u_0 - v_0\|_{\mathbf{X}} + \overline{M_A} \int_0^t \sup_{0 \le s \le T} \|f(s, u(s)) - \hat{f}(s, u(s))\| ds \\
&\quad + \overline{M_A} K \int_0^t \|u(s) - v(s)\| ds \\
&\le \overline{M_A}(\epsilon_1 T + \epsilon_2) + \overline{M_A} K \int_0^t \|u(s) - v(s)\| ds
\end{aligned}$$

By Gronwall's lemma , we have that

$\|u(t) - v(t)\|_{\mathbf{X}} \le \overline{M_A}(\epsilon_1 T + \epsilon_2) e^{\overline{M_A} K}, \ 0 \le t \le T$. Proving the proposition.

Now consider the following hypothesis
(H_{3b}): There exists $K_f, K_{\hat{f}} \in \mathcal{C}([0, T]; (0, \infty))$ such that f and $\hat{f}$ satisfy

$\|f(t,x) - f(t,y)\| \leq K_f(t)\|x-y\|, \ \forall x,y \in \mathbf{X}, \forall t \in [0,T].$

In the proposition above let's replace (H_3) by (H_{3b}) and assume that (H_A) and $(H_{3b}) - (H_5)$ hold. Then we have that

$$\begin{aligned}
\|u(t)-v(t)\|_{\mathbf{X}} &\leq \|e^{tA}\|\|u_0 - v_0\|_{\mathbf{X}} + \int_0^t \|e^{(t-s)A}\|\|f(s,u(s)) - \hat{f}(s,v(s))\|ds \\
&\leq \overline{M_A}\|u_0 - v_0\|_{\mathbf{X}} + \int_0^t \overline{M_A}\|f(s,u(s)) - \hat{f}(s,u(s)) + \hat{f}(s,u(s)) - \hat{f}(s,v(s))\|ds \\
&\leq \overline{M_A}\|u_0 - v_0\|_{\mathbf{X}} + \overline{M_A}\int_0^t \sup_{0\leq s\leq T} \|f(s,u(s)) - \hat{f}(s,u(s))\|ds + \\
&\quad \overline{M_A}\int_0^t K_{\hat{f}}(s)\|u(s)-v(s)\|ds \\
&\leq \overline{M_A}(\epsilon_1 T + \epsilon_2) + \overline{M_A}\int_0^t K_{\hat{f}}(s)\|u(s)-v(s)\|ds
\end{aligned}$$

By Gronwall's lemma, we have that

$$\|u(t)-v(t)\|_{\mathbf{X}} \leq \overline{M_A}(\epsilon_1 T + \epsilon_2)e^{\overline{M_A}\int_0^T K_{\hat{f}}(s)ds} \leq \overline{M_A}(\epsilon_1 T + \epsilon_2)e^{\overline{M_A}KT}$$

since $K_{\hat{f}} \in \mathcal{C}([0,T];(0,\infty))$ and $[0,T]$ is closed and bounded, there exists a constant K such that $K_{\hat{f}}(t) \leq K, \forall t \in [0,T]$.
Hence we still have a continuous dependence result established.

3.2.4 Extendability of Local Solutions

Definition 3.2.1 *Let u be a local mild solution of the Cauchy problem (3.1) u is called a maximal solution if it can not be extended on a larger time interval.*

In this section we see how and when a local mild solution can be extended to a larger time interval and possibly to the whole interval $[0,T]$.

Strategy P: One approach could be to use the point T_0 as initial condition if

$$\lim_{t\to T_0^-}(\phi u)(t)$$

exists in the following IVP (since the solution map ϕ defined in proposition 3.2.1 is continuous at $t = T_0$)

$$\begin{cases} \bar{u}'(t) &= A\bar{u}(t) + f(t,\bar{u}(t)), \quad t > T_0 \\ \bar{u}(0) &= \lim_{t\to T_0^-}(\phi u)(t) \end{cases} \tag{3.22}$$

Applying the same method again as in the local existence result (proposition 3.2.1) we could obtain a mild solution of the above IVP on $[T_0, T_0+\epsilon)$ and construct a piece-wise defined function $V : [0, T_0+\epsilon) \longrightarrow \mathbf{X}$ by

$$V(t) = \begin{cases} u(t), & 0 \le t < T_0 \\ \bar{u}(t), & T_0 \le t < T_0 + \epsilon \end{cases} \tag{3.23}$$

V is a mild solution of (3.1) on $[0, T_0 + \epsilon)$ since it is continuous. This continuity is obvious since from the expression of the solution u and the definition of $\bar{u}$, we have
$\lim\limits_{t \to T_0^+} V(t) = \bar{u}(T_0) = \lim\limits_{t \to T_0^-} u(t) = \lim\limits_{t \to T_0^-} V(t)$.
But this approach does not work all the time since sometimes the nature of the solution itself makes it difficult to obtain a maximal mild solution, as we can see in the following example.

Example: Consider the following IVP.

$$\begin{cases} x'(t) &= (1 + 2x(t))^4, \quad t > 0 \\ x(0) &= x_0 \end{cases} \tag{3.24}$$

a) We easily verify that for $x_0 \neq -\frac{1}{2}$ a solution of the above IVP is given by

$$x(t) = \frac{1}{2}[-1 + ((2x_0 + 1)^{-3} - 6t)^{-\frac{1}{3}}] \tag{3.25}$$

b) Let $T = \frac{1}{6}(2x_0 + 1)^{-3}$ and assume that $x_0 > -\frac{1}{2}$. Then we have that :

$\lim\limits_{t \to T^-} \mid x(t) \mid = \lim\limits_{t \to T^-} \frac{1}{2}[-1 + \frac{1}{\sqrt[3]{(2x_0+1)^{-3} - 6t}}] = +\infty$

Thus $\lim\limits_{t \to T^-} \mid x(t) \mid = +\infty$
So (3.24) does not have a global mild solution on the interval $[0, \frac{1}{6}(2x_0 + 1)^{-3}]$. The presence of the vertical asymptote $t = \frac{1}{6}(2x_0+1)^{-3}$ makes it impossible to get a global solution on $[0, \frac{1}{6}(2x_0+1)^{-3}]$.

In finding the maximal time interval $[0, T_{max})$ to which a mild solution can be extended, we would need the following result with conditions involving initial data that tell us when a mild solution is globally defined. First a definition of global solution.

Theorem 3.2.3 (Extendability of a Mild Solution) *Assume that (H_A) and (H_2) hold. Then there exists $T_{max} \in (0, +\infty]$ with the following properties:*
There exists $u \in \mathcal{C}([0, T_{max}); \mathbf{X})$ such that for all $0 < T < T_{max}$, u is the unique solution of (3.5) on $[0, T]$. In addition

$$2M(\|f(.,0)\|_{+\infty} + 2\|u(t)\|) \geq \frac{1}{T_{max} - t} - 2, \forall t \in [0, T_{max}) \tag{3.26}$$

(where M is the local Lipschitz constant). In particular we have the following alternatives.
i) $T_{max} = +\infty$
ii) $T_{max} < +\infty$ or $\lim\limits_{t \to T_{max}^-} \|u(t)\| = +\infty$

Remark: If property (i) above holds , we say that the solution u is global. On the other hand if (ii) holds, we say that u blows up in finite time, as in the example above.

Proof of the Theorem
Observe that if we suppose $T_{max} < \infty$, then

$\|u(t)\| \geq \frac{1}{4M(T_{max}-t)} - \frac{1}{2M} - \frac{\|f(.,0)\|}{2}$

$\Rightarrow \limsup_{t\to T_{max}^-}\|u(t)\| \geq +\infty \quad \Rightarrow \lim_{t\to T_{max}^-} \|u(t)\| = +\infty.$
ie $T_{max} < +\infty \Rightarrow \lim_{t\to T_{max}^-} \|u(t)\| = +\infty$. so set

$T_{max} = \sup\{T > 0 : \exists u \in \mathcal{C}([0,T], \mathbf{X}) \quad solution \quad of \quad (3.5)\}$

By the local existence result (proposition 3.2.1), we know that $T_{max} > 0$. By the approach in *strategy P* and the uniqueness of the mild solution in proposition 3.2.1 we can build a maximal solution $u \in \mathcal{C}([0,T_{max}); \mathbf{X})$ of (3.5). It remains now to show inequality (3.26). This inequality is immediate if $T_{max} = +\infty$. So we may assume that $T_{max} < +\infty$
We argue by contradiction.
Assume that there exists $t_0 \in [0,T_{max})$ such that (3.26) does not hold.

Let $T_L = \frac{1}{2M(2L+\|f(.,0)\|)+2} > 0, \ with \quad L = \|u(t_0)\|$

Then we have that $T_{max} - t_0 < T_L$
Let $v \in \mathcal{C}([0,T_L]; \mathbf{X})$ be the solution given by (3.5) of

$v(s) = e^{sA}u(t) + \int_0^s e^{(s-\tau)A} f(\tau, v(\tau))d\tau, \forall s \in [0,T_L]$

Then we define $w \in \mathcal{C}([0,t_0+T_L]; \mathbf{X})$ by

$$w(t) = \begin{cases} u(s), & 0 \leq s \leq t_0 \\ v(s-t_0), & t_0 \leq s \leq t_0 + T_L \end{cases} \tag{3.27}$$

Taking $T = t_0 + T_L$, we have that , w is a solution of (3.5) on $[0,T]$, which contradicts the definition of T_{max} since $T_{max} < t_0 + T_L$. Hence there exists $t \in [0,T_{max})$ such that (3.26) holds. And this completes the proof of the theoerem.

3.2.5 Global Existence of Solutions

Proposition 3.2.3 *Let $R > 0$. Suppose that (H_A) holds and that $f : [0,T] \times \mathcal{B}_{\mathbf{X}}(u_0, R) \to \mathbf{X}$ is Lipschitzian with respect to the second variable on the ball $\mathcal{B}_{\mathbf{X}}(u_0, R)$ (uniformly in t), and assume in addition that for some $0 < R^* \leq R$*

$$\|u(t) - u_0\|_{\mathbf{X}} \leq R^*, \ \forall t \in [0,T]$$

where u is a mild solution of (3.1), guaranteed by Proposition 3.2.1. Then u is global and the unique mild solution of (3.1) on $[0,T]$. (That is (3.1) has a unique global mild solution on $[0,T]$)

Proof
Let $[0,T_0)$ be the maximal interval of definition of u. Then by the Lipschitzness of f, there exists $L_0 > 0$ such that:

$$\|f(t,x)-f(t,y)\|_{\mathbf{X}} \le L_0\|x-y\|_{\mathbf{X}},\ \forall x,y \in \mathcal{B}_{\mathbf{X}}(u_0,R),\ 0\le t\le T$$

And so

$$\|f(s,x)\| \ \le\ \|f(\cdot,0)\|_\infty + L_0\|x\|,\ \forall x \in \mathcal{B}_{\mathbf{X}}(u_0,R),\ 0\le s\le T$$

Then it is not hard to see that u is bounded by using Gronwall's Lemma. So there exists $R_0>0$ such that $\|u(t)\| \le R_0,\ \forall t\in[0,T]$. Also it follows that u is uniformly continuous on $[0,T_0)$ and therefore can be extended by continuity to $[0,T_0]$. Set $u(T_0) = \lim\limits_{t\to T_0^-} u(t)$.

We show that $T_0 = T$. Suppose by contradiction that $T_0 < T$ and let us consider the problem

$$\left\{\begin{array}{rcl} y'(t) &=& Ay(t)+f(t,y(t)),\quad 0<t<T_0 \\ y(0) &=& u(T_0) \end{array}\right. \tag{3.28}$$

This problem has a unique mild solution y by Proposition 3.2.1. Therefore it is not hard to see that the function

$$u^*(t) \quad = \quad \left\{\begin{array}{ll} u(t), & 0\le t<T_0 \\ y(t), & T_0\le t<T_0+\epsilon \end{array}\right.$$

is a mild solution of (3.1) on $[0,T_0+\epsilon)$ (for some $\epsilon>0$), contradicting the fact that $[0,T_0)$ is a maximal interval of solution. Hence u is the unique global mild solution on $[0,T]$.

Proposition 3.2.4 *Assume* $(H_A),(H_2)$ *and that there exist positive constants* C_1, C_2 *such that* $\|f(t,x)\|_{\mathbf{X}} \le C_1\|x\|_{\mathbf{X}} + C_2, \forall t>0, x\in\mathbf{X}$
Then, for all $T>0$ *(3.1) has a unique mild solution on* $[0,T]$.

Proof: By the extendability theorem, (3.1) has a unique mild solution u on $[0,\,T_{max})$ expressed by (3.5).
Suppose $T_{max} < +\infty$, then

$$\sup_{0\le t\le T_{max}} \|u(t)\| \le Me^{\alpha T_{max}}(\|u_0\| + \textstyle\int_0^{T_{max}}[k_0R + \|f(s,u_0)\|]ds < +\infty$$

So $\exists K>0$ *such that* $\|u\|\le K$. Therefore u is bounded on $[0,T_{max})$. Hence the limit of $u(t)$ as $t\to T_{max}$ exists in X, which contradicts the maximality of the interval. Hence $T_{max}=+\infty$. Therefore (3.0.1) has a unique mild solution on $[0,+\infty)$.

3.2.6 Long-Term Behavior of Solutions

In this section we assume that $T_{max}=+\infty$ and that a unique mild solution u exists on $[0,+\infty)$. To examine the long-term behavior of this unique solution , we need to answer the following questions:
i) Does there exists $v\in\mathbf{X}$ such that $\lim\limits_{t\to+\infty} u(t) = v$?
ii) Do we have $\lim\limits_{t\to+\infty}\|u(t)\|_{\mathbf{X}} < +\infty$?
iii) Is the solution time-periodic? ie does there exists a non zero $P\in\mathbb{R}$ such that $u(t+P)=u(t)$, when u is a global solution?

Consider (3.1). **Suppose** (H_A) **holds and that the semigroup is contractive with** $w<0$

such that $\|e^{tA}\| \leq e^{wt}$.
We have the following results.

1)***Suppose moreover that*** f ***is globally Lipschitz (uniformly in*** t ***on*** $[0, +\infty)$***).***
Then $\lim_{t \to +\infty} \|u(t)\|_{\mathbf{X}} = 0$

Proof: We have

$$
\begin{aligned}
\|u(t)\|_{\mathbf{X}} &\leq \|e^{tA}u_0\|_{\mathbf{X}} + \int_0^t \|e^{(t-s)A}\| \|f(s, u(s))\| ds \\
&\leq \|e^{tA}u_0\|_{\mathbf{X}} + \int_0^t \|e^{(t-s)A}\| \|f(s, u(s)) - f(s, 0) + f(s, 0)\| ds \\
&\leq e^{wt}\|u_0\|_{\mathbf{X}} + k_0 e^{wt} \int_0^t \|u(s)\| ds + e^{wt}\|f(., 0)\|_{\mathbf{L}^1} \\
&\leq e^{wt}(\|u_0\|_{\mathbf{X}} + \|f(., 0)\|_{\mathbf{L}^1}) + k_0 e^{wt} \int_0^t \|u(s)\| ds
\end{aligned}
$$

By Gronwall's lemma, we have

$\|u(t)\|_{\mathbf{X}} \leq e^{wt}(\|u_0\|_{\mathbf{X}} + \|f(., 0)\|_{\mathbf{L}^1})e^{k_0 t e^{wt}} \longrightarrow 0 \quad as \quad t \longrightarrow +\infty$ since $w < 0$ by assumption and $\lim_{t \to +\infty} te^{wt} = 0$. Thus $\lim_{t \to +\infty} \|u(t)\|_{\mathbf{X}} = 0$ as desired.

2) ***Assume (3.1) has a mild solution*** u ***on*** $[0, +\infty)$ ***and*** f ***satisfies the sublinear growth:*** $\|f(t, x)\|_{\mathbf{X}} \leq C_1\|x\| + C_2, \forall t > 0, x \in \mathbf{X}$. ***Then*** $\lim_{t \to +\infty} \|u(t)\|_{\mathbf{X}} = 0$

Proof: we have

$$
\begin{aligned}
\|u(t)\|_{\mathbf{X}} &\leq \|e^{tA}u_0\|_{\mathbf{X}} + \int_0^t \|e^{(t-s)A}\| \|f(s, u(s))\| ds \\
&\leq e^{wt}\|u_0\|_{\mathbf{X}} + e^{wt} \int_0^t [C_1\|u(s)\| + C_2] ds \\
&\leq e^{wt}(\|u_0\|_{\mathbf{X}} + C_2 t) + C_1 e^{wt} \int_0^t \|u(s)\| ds
\end{aligned}
$$

By Gronwall's lemma, we have

$\|u(t)\|_{\mathbf{X}} \leq (e^{wt}\|u_0\|_{\mathbf{X}} + C_2 t e^{wt})e^{C_1 t e^{wt}} \longrightarrow 0 \quad as \quad t \longrightarrow +\infty$. Hence $\lim_{t \to +\infty} \|u(t)\|_{\mathbf{X}} = 0$ as desired

3.3 Theory for Non-Lipschitz-Type Forcing Terms

3.3.1 Existence and Uniqueness Result

Consider the following sequence of successive approximations of mild solution of (3.1)

$u_n(t) = e^{tA}u_0 + \int_0^t e^{(s-t)A} f(s, u_{n-1}(s))ds, \forall t \in [0, T]$ (6)

In this section we consider (3.1) where the forcing term satisfies the following hypothesis: $f : [0, T] \times \mathbf{X} \longrightarrow \mathbf{X}$ is such that.

(H_7) ***There exists*** $K_1 : [0, T] \times [0, +\infty) \longrightarrow [0, +\infty)$ ***such that***
i) $K_1(., x)$ ***is integrable for all*** $x \in [0, +\infty)$
ii) $K_1(t, .)$ ***is continuous ,nondecreasing and concave*** $\forall t \in [0, T]$
iii) $\|f(t, x)\|_{\mathbf{X}} \le K_1(t, \|x\|_{\mathbf{X}}), \forall t \in [0, T]\ x \in \mathbf{X}$

(H_8) ***There exists*** $K_2; [0, T] \times [0, +\infty) \longrightarrow [0, +\infty)$ ***such that***
i) $K_2(., x)$ ***is integrable*** $\forall x \in [0, +\infty)$
ii) $K_2(t, .)$ ***is continuous ,nondecreasing and*** $K_2(t, 0) = 0, \forall t \in [0, T]$
iii) $\|f(t, x) - f(t, y)\|_{\mathbf{X}} \le K_2(t, \|x - y\|_{\mathbf{X}}), \forall t \in [0, T], x, y \in \mathbf{X}$

(H_9) ***Any function*** $w : [0, T] \longrightarrow [0, +\infty)$ ***which is continuous,*** $w(0) = 0$ ***and satisfies*** $w(t) \le \overline{M_A} \int_0^t K_2(s, w(s))ds,\ 0 \le t \le T^* \le T$ ***must be identically zero on*** $[0, T^*]$

We shall need the following result.

Lemma 3.3.1 *For every* $\beta_1, \beta_2 > 0, \exists\ 0 < T_1 \le T$ *such that the equation*

$$z(t) = \beta_1 + \beta_2 \int_0^t K_1(s, z(s))ds \tag{3.29}$$

has a continuous local solution $z : [0, T_1) \longrightarrow [0, +\infty)$

We now state and prove the existence and uniqueness result of this section.

Theorem 3.3.1 *Assume* $(H_A), (H_7) - (H_9)$ *hold. Then there exists* $0 < T^* \le T$ *such that (3.1) has a unique local mild solution* $u : [0, T^*] \longrightarrow \mathbf{X}$

Proof
We use the successive approximations method
For any $\beta_1 > \overline{M_A}\|u_0\|_{\mathbf{X}}$, lemma 3.3.1 guarantees the existence of $0 < T_1 \le T$ for which the equation

$$z(t) = \beta_1 + \overline{M_A} \int_0^t K_1(s, z(s))ds$$

has a unique solution $z : [0, T_1) \longrightarrow [0, +\infty)$. We divide the proof into many claims.
<u>Claim 1</u>: For each $n \in \mathbb{N}, \|u_n(t)\|_{\mathbf{X}} \le z(t),\ 0 \le t < T_1 \le T$

We prove this by induction on n. For $n = 1$ and $0 \leq t < T_1$ we have

$$\begin{aligned}
\|u_1(t)\|_{\mathbf{X}} &\leq \|e^{tA}u_0\|_{\mathbf{X}} + \int_0^t \|e^{(t-s)A}\|\|f(s,u_0)\|ds \\
&\leq \overline{M_A}\|u_0\|_{\mathbf{X}} + \overline{M_A}\int_0^t K_1(s,\|u_0\|)ds \\
&\leq \overline{M_A}\|u_0\|_{\mathbf{X}} + \overline{M_A}\int_0^t K_1(s,\overline{M_A}\|u_0\|)ds \\
&\leq \overline{M_A}\|u_0\|_{\mathbf{X}} + \overline{M_A}\int_0^t K_1(s,\beta_1)ds \\
&\qquad \textit{since} \quad \overline{M_A}\|u_0\| \leq \beta_1, \quad \textit{and} \quad K(t,.) \quad \textit{is} \quad \textit{increasing} \\
&\leq \beta_1 + \overline{M_A}\int_0^t K_1(s,z(s))ds \\
&= z(t)
\end{aligned}$$

So $\|u_1(t)\|_{\mathbf{X}} \leq z(t), \forall t \in [0,T_1)$

Now assume that the claim holds up to order n. ie $\|u_n(t)\|_{\mathbf{X}} \leq z(t), \forall t \in [0,T_1)$

We have that

$$\begin{aligned}
\|u_{n+1}(t)\|_{\mathbf{X}} &\leq \|e^{tA}u_0\|_{\mathbf{X}} + \int_0^t \|e^{(t-s)A}\|\|f(s,u_n(s))\|ds \\
&\leq \overline{M_A}\|u_0\|_{\mathbf{X}} + \overline{M_A}\int_0^t K_1(s,\|u_n(s)\|)ds \\
&\leq \overline{M_A}\|u_0\|_{\mathbf{X}} + \overline{M_A}\int_0^t K_1(s,z(s))ds \quad \textit{by} \quad \textit{induction} \quad \textit{hypothesis} \\
&\leq \beta_1 + \overline{M_A}\int_0^t K_1(s,z(s))ds \\
&= z(t)
\end{aligned}$$

Hence $\|u_{n+1}(t)\|_{\mathbf{X}} \leq z(t), \forall t \in [0,T_1)$ and therefore this proves the claim.

Claim 2: For every $\delta_0 > 0, \exists 0 < T_2 \leq T_1$ ***such that*** $\forall n \in \mathbb{N}$

$$\|u_n(t) - e^{tA}u_0\| \leq \delta_0, \quad 0 \leq t < T_2 \leq T_1 \leq T$$

Proof: By induction on n. let $\delta_0 > 0$ be fixed. For $n = 1$ we have.

$$\begin{aligned}
\|u_1(t) - e^{tA}u_0\|_{\mathbf{X}} &\leq \int_0^t \|e^{(t-s)A}\|\|f(s,u_0)\|ds \\
&\leq \overline{M_A}\int_0^t K_1(s,\|u_0\|)ds \\
&\leq \overline{M_A}\int_0^t K_1(s,\overline{M_A}\|u_0\|)ds \\
&\leq \overline{M_A}\int_0^t K_1(s,\beta_1)ds, \quad since \quad \overline{M_A}\|u_0\| \leq \beta_1, \quad and \quad K(t,.) \quad is \quad increasing \\
&\leq \overline{M_A}\int_0^t K_1(s,z(s))ds
\end{aligned}$$

We have that z is continuous and K_1 is integrable, this implies that the map $t \mapsto \overline{M_A}\int_0^t K_1(s,z(s))ds$ is absolutely continuous. Therefore there exists $0 < T_2 \leq T_1$ such that

$\overline{M_A}\int_0^t K_1(s,z(s))ds \leq \delta_0,\ 0 < T_2 \leq T_1$ (7).

Thus $\|u_1(t) - e^{tA}u_0\| \leq \delta_0,\ 0 < T_2 \leq T_1$.
Assume that the claim holds up to order n. Then we have

$$\begin{aligned}
\|u_{n+1}(t) - e^{tA}u_0\|_{\mathbf{X}} &\leq \int_0^t \|e^{(t-s)A}\|\|f(s,u_n)\|ds \\
&\leq \overline{M_A}\int_0^t K_1(s,\|u_n\|)ds \\
&\leq \overline{M_A}\int_0^t K_1(s,z(s))ds, \quad by \quad induction \quad hypothesis \\
&\leq \delta_0, \quad by \quad (7)
\end{aligned}$$

Hence for all $\delta_0 > 0$ and $n \in \mathbb{N}, \exists 0 < T_1 \leq T_2$ such that
$\|u_n(t) - e^{tA}u_0\| \leq \delta_0,\ 0 \leq t < T_1 \leq T_2 \leq T$

Claim 3: For every $n, m \in \mathbb{N}, \|u_{n+m}(t) - u_n(t)\|_{\mathbf{X}} \leq \overline{M_A}\int_0^t K_2(s,2\delta_0)ds, 0 \leq t < T_2$
Proof

Let $n, m \in \mathbb{N}$. We have that, for $0 \leq t < T_2 \leq T_1 \leq T$

$$\begin{aligned}
\|u_{n+m}(t) - u_n(t)\|_{\mathbf{X}} &\leq \int_0^t \|e^{(t-s)A}\| \|f(s, u_{n+m-1}(s) - f(s, u_{n-1}(s))\| ds \\
&\leq \overline{M_A} \int_0^t K_2(s, \|u_{n+m-1}(s) - u_{n-1}(s)\|) ds \\
&\leq \overline{M_A} \int_0^t K_2(s, \|u_{n+m-1}(s) - e^{sA}u_0\| + \|e^{sA}u_0 - u_{n-1}(s)\|) ds \\
&\leq \overline{M_A} \int_0^t K_2(s, 2\delta_0) ds \\
&\quad since \quad \|u_{n+m-1}(s) - e^{sA}u_0\| \leq \delta_0 \\
&\quad by \quad claim \quad 2 \quad and \quad monotonicity \quad of \quad K_2
\end{aligned}$$

Thus $\|u_{n+m}(t) - u_n(t)\|_{\mathbf{X}} \leq \overline{M_A} \int_0^t K_2(s, 2\delta_0) ds$ for all $n, m \in \mathbb{N}$ proving the claim.

Now define the following maps.
$\gamma_n : [0, T_2] \longrightarrow (0, +\infty) \quad and \quad \theta_{m,n} : [0, T_2] \longrightarrow (0, +\infty)$ by

$\gamma_1(t) = \overline{M_A} \int_0^t K_2(s, 2\delta_0) ds, \quad \gamma_n(t) = \overline{M_A} \int_0^t K_2(s, \gamma_{n-1}(s)) ds, \ n \geq 2$

$\theta_{m,n} = \|u_{n+m}(t) - u_n(t)\|_{\mathbf{X}}, m, n \in \mathbb{N}$
The integrability of K_2 implies the absolute continuity of the integral $\overline{M_A} \int_0^t K_2(s, 2\delta_0) ds$, there exists $0 < T_3 \leq T_2$ such that $\gamma_1(t) \leq 2\delta_0, \forall t \in [0, T_3)$

Claim 4: For every $n \geq 2$,
$\gamma_n(t) \leq \gamma_{n-1}(t) \leq .... \leq \gamma_1(t), \forall t \in [0, T_3)$
Proof We do this by induction on n. For $n = 2$ we have that $\forall t \in [0, T_3)$

$$\begin{aligned}
\gamma_2(t) &= \overline{M_A} \int_0^t K_2(s, \gamma_1(t)) ds \\
&\leq \overline{M_A} \int_0^t K_2(s, 2\delta_0) ds \quad by \quad monotonicity \quad of \quad K_2, \quad since \quad \gamma_1(t) \leq 2\delta_0 \\
&= \gamma_1(t)
\end{aligned}$$

Now assume that the claim holds up to order n, Then it follows that $\forall t \in [0, T_3)$
$\gamma_{n+1}(t) = \overline{M_A} \int_0^t K_2(s, \gamma_n(s)) ds \leq \overline{M_A} \int_0^t K_2(s, \gamma_{n-1}(s)) ds = \gamma_n(t)$

ie $\gamma_{n+1}(t) \leq \gamma_n(t)$, proving the claim.

Claim 5: For every $n, m \in \mathbb{N}$, $\theta_{m,n}(t) \leq \gamma_n(t), \ \forall t \in [0, T_3)$
proof: We do this by induction. For $n = 1$, note that $\forall m \in \mathbb{N}, \ and \ 0 \leq t < T_3$, claim 3 implies that:

$\theta_{m,1} = \|u_{m+1}(t) - u_1(t)\|_{\mathbf{X}} \leq \overline{M_A} \int_0^t K_2(s, 2\delta_0)ds = \gamma_1(t)$

Next assume that the claim holds up to n, uniformly $\forall m \in \mathbb{N}$.
Observe that $\forall m \in \mathbb{N}$ *and* $0 \leq t < T_3$

$$\begin{aligned}
\theta_{m,(n+1)} &= \|u_{m+n+1}(t) - u_{n+1}(t)\|_{\mathbf{X}} \\
&\leq \overline{M_A} \int_0^t \|f(s, u_{m+n}(s)) - f(s, u_n(s))\| ds \\
&\leq \overline{M_A} \int_0^t K_2(s, \|u_{m+n}(s) - u_n(s)\|) ds \\
&\leq \overline{M_A} \int_0^t K_2(s, \theta_{m,n}(s)) ds \\
&\leq \overline{M_A} \int_0^t K_2(s, \gamma_n(s)) ds \quad \textit{since} \quad \theta_{m,n}(t) \leq \gamma_n(t) \\
&= \gamma_{n+1}(t)
\end{aligned}$$

ie $\theta_{m,(n+1)}(t) \leq \gamma_{n+1}(t)$,and this proves the theorem.

Let $0 \leq t < T < T_3$ and $n_0 \in \mathbb{N}$ be fixed. Then we have that $\gamma_{n_0}(t) \leq \gamma_{n_0}(T)$.

Claim 6: There exists $u \in \mathcal{C}([0, T_3); \mathbf{X})$ ***such that*** $\lim_{n \to +\infty} \|u_n - u\|_{\mathcal{C}([0,T_3);\mathbf{X})} = 0$
Proof: Define $\gamma : [0, T_3) \longrightarrow \mathbb{R}$ by $\gamma(t) = \lim_{n \to +\infty} \gamma_n(t) = \inf_{n \in \mathbb{N}} \gamma_n(t)$. γ is well-defined since the sequence $\{\gamma_n(t)\}_{n \geq 1}$ is decreasing.
We show that γ is non negative and $\gamma(0) = 0$
We have that for $n \in \mathbb{N}$ fixed
$\gamma_n(0) \leq \gamma_(t)$ since $\gamma_n(t)$ is increasing in t
$\Rightarrow 0 \leq \gamma_n(t)$ since $\gamma_n(0) = 0$
$\Rightarrow 0 \leq \lim_{n \to +\infty} \gamma_n(t)$
$\Rightarrow 0 \leq \gamma(t)$
Also $\gamma(0) = 0$ since $\gamma(0) = \lim_{n \to +\infty} \gamma_n(0) = 0$
Continuity of γ
By the non-increasing property of $\{\gamma_n(t)\}_{n \geq 1}$ and the monotonicity of K_2, we have that

$$K_2(s, \gamma_n(s)) \leq K_2(s, \gamma_1(s)) \leq K_2(s, 2\delta_0)$$

Also $K_2(s, .)$ is continuous.
So

$$\lim_{n \to +\infty} K_2(s, \gamma_n(s)) = K_2(s, \lim_{n \to +\infty} \gamma_n(s))$$

It follows that.

$$\begin{aligned}
\gamma(t) &= \lim_{n\to+\infty}\gamma_{n+1}(t) \\
&= \lim_{n\to+\infty}\overline{M_A}\int_0^t K_2(s,\gamma_n(s))ds \\
&= \overline{M_A}\int_0^t \lim_{n\to+\infty}K_2(s,\gamma_n(s))ds \quad \textit{by Lebesgue Dominated Convergence} \\
&= \overline{M_A}\int_0^t K_2(s,\lim_{n\to+\infty}\gamma_n(s))ds \\
&= \overline{M_A}\int_0^t K_2(s,\gamma(s))ds
\end{aligned}$$

So $\gamma(t) = \overline{M_A}\int_0^t K_2(s,\gamma(s))ds$. Since K_2 is continuous and integrable, the integral is strongly continuous. Therefore γ is a continuous map and moreover

$\gamma(t) \le \overline{M_A}\int_0^t K_2(s,\gamma(s))ds, \forall t\in[0,T_3)$. Hence by (H_9), $\gamma(t)=0, \forall t\in[0,T_3)$.
Further, using claim 5 and the increasing property of $\gamma_n(t)$ (for each fixed n) in t, we have that:

$$\|u_{m+n}-u_n\|_{\mathcal{C}([0,T_3);\mathbf{X})} = \sup_{0\le t<T_3}\theta_{m,n}(t) \le \sup_{0\le t<T_3}\gamma_n(t)\le\gamma_n(T_3)$$

Letting $n\longrightarrow+\infty$, we have

$$\lim_{n\to+\infty}\|u_{m+n}-u_n\|_{\mathcal{C}([0,T_3);\mathbf{X})} \le \lim_{n\to+\infty}\gamma_n(T_3)=\gamma(T_3)=0$$

Consequently $\{u_n\}_{n\ge1}$ is a Cauchy sequence in $\mathcal{C}([0,T_3);\mathbf{X})$
and hence converges to some $u\in\mathcal{C}([0,T_3);\mathbf{X})$, proving the claim.

Claim 7: The function u from claim 6 is a mild solution of (3.1) on $[0,T_3)$
Proof: Let $y(t)=e^{tA}u_0+\int_0^t e^{(t-s)A}f(s,u(s))ds$.

We must argue that $\lim_{n\to+\infty}\|u_n-y\|_{\mathcal{C}([0,T_3);\mathbf{X})}=0$.
To this end observe that

$$\begin{aligned}
\|u_n(t)-y(t)\|_{\mathbf{X}} &= \|\int_0^t e^{(t-s)A}[f(s,u_{n-1}(s))-f(s,u(s))]ds\| \\
&= \overline{M_A}\int_0^t K_2(s,\|u_{n-1}(s)-u(s)\|)ds \\
&= \overline{M_A}\int_0^t K_2(s,\sup_{0\le s<T_3}\|u_{n-1}(s)-u(s)\|)ds \\
&= \overline{M_A}\int_0^t K_2(s,\|u_{n-1}-u\|_{\mathcal{C}([0,T_3);\mathbf{X})})ds
\end{aligned}$$

Taking the supremum over $[0,T]$ yields

$\|u_n - y\|_{\mathcal{C}([0,T_3);\mathbf{X}} \leq \overline{M_A} \int_0^t K_2(s, \|u_{n-1} - u\|_{\mathcal{C}([0,T_3);\mathbf{X}})ds$. Taking the limit as $n \longrightarrow +\infty$ and using claim 6, the continuity of $K_2(t,\cdot)$ and the Lebesgue Dominated Convergence, we have

$0 \leq \lim_{n\to+\infty} \|u_n - y\|_{\mathcal{C}([0,T_3);\mathbf{X})} \leq \overline{M_A} \int_0^t K_2(s,0)ds = 0.$

Thus $\lim_{n\to+\infty} \|u_n - y\|_{\mathcal{C}([0,T_3);\mathbf{X})} = 0$ and therefore $u = y$ by unique of limit. This proves the claim.

Claim 8: The mild solution of (3.1) on $[0,T_3)$ ***from claim 7 is unique.***
Proof: let $x, y \in \mathcal{C}([0,T_3);\mathbf{X}$ be two mild solutions of (3.1). Substracting their variation of parameter formula, we get

$$\begin{aligned} \|x(t) - y(t)\|_{\mathbf{X}} &\leq \overline{M_A} \int_0^t \|f(s,x(s)) - f(s,y(s)\|ds \\ &\leq \overline{M_A} \int_0^t K_2(s, \|x(s) - y(s)\|_{\mathbf{X}})ds, \quad 0 \leq t < T_3 \end{aligned}$$

But the map $t \mapsto \|x(t) - y(t)\|_{\mathbf{X}}$ is continuous and $\|x(0) - y(0)\|_{\mathbf{X}} = 0$
So by (H_9), we have $\|x(t) - y(t)\|_{\mathbf{X}} = 0, \forall t \in [0,T_3)$. Thus the mild solution is unique. This proves the claim and therefore completes the proof of the theorem.

3.3.2 Theory under compactness assumption

In this section, we investigate (3.1) in the case where the forcing term f has weaker conditions that guarantee existence but do no longer ensure uniqueness of a mild solution of (3.1). In application the forcing term arising may be only continuous and sometimes less regular.

Definition 3.3.1 *A $\mathcal{C}_0$-semigroup $\{e^{tA} : t \geq 0\}$ is compact if $e^{tA} : \mathbf{X} \mapsto \mathbf{X}$ is a compact operator , $\forall t \in (0,T)$*

Example: If $A \in \mathcal{M}^N(\mathbb{R})$ then $\{e^{tA} : 0 < t \leq T\}$ is a compact operator on $\mathbb{R}^N$ since $\mathbb{R}^N$ is finite dimensional and $e^{tA} : \mathbb{R}^N \to \mathbb{R}^N$ is bounded.
Now consider the following hypothesis which we will use in the subsequent sections.

$$(H_A^*) : \textbf{ A generates a compact semigroup } \{e^{tA} : t \geq 0\} \textbf{ on } \mathbf{X}.$$

We now state one of the most important result of this section that gives us the existence of a mild solution of (3.1).

Theorem 3.3.2 *Let A:$\mathcal{D}(A) \subset \mathbf{X} \to \mathbf{X}$ be a linear operator. Assume that A is the infinitesimal generator of a $\mathcal{C}_0$-semigroup $\{e^{tA} : t \geq 0\}$ such that e^{tA} is compact for all $t \in (0,T]$ and*
(H_2) f:$[0,T] \times \mathbf{X} \to \mathbf{X}$ is a continuous map for which there exist $K_1, K_2 \in \mathcal{C}([0,T];\mathbb{R})$ such that

$$\|f(t,x)\| \leq K_1(t)\|x\| + K_2(t), \;\; \forall t \in [0,T], \forall x \in \mathbf{X}$$

.

Then the Cauchy problem (3.1) has at least one global mild solution u on $[0,T]$.

Proof: We use Schaefer's fixed-point theorem. Consider the solution map ϕ defined by:

$$\phi : \mathcal{C}([0,T];\mathbf{X}) \to \mathbf{X} \ , \ (\phi u)(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A} f(s, u(s))ds$$

We devide the proof into three steps.
Step 1: We show that ϕ is continuous.

Proof: Let $\{v_n\} \subset \mathcal{C}([0,T];\mathbf{X})$ be such that $\lim_{n\to\infty} \|v_n - v\| = 0$.
We must show that $\lim_{n\to\infty} \|\phi(v_n) - \phi(v)\| = 0$. Note that for all $0 \le s \le T$ and $\alpha > 0$, we have

$$\begin{aligned}
\left\|\frac{f(s,v_n(s)) - f(s,u(s))}{e^{\alpha s}}\right\| &\le \|f(s,v_n(s)) - f(s,v(s))\| \quad since \quad e^{\alpha s} \ge 1 \\
&\le \|f(s,v_n(s))\| + \|f(s,v(s))\| \\
&\le \|K_1\|\|v_(s)\| + \|K_2\| + \|K_1\|\|v(s)\| + \|K_2\| \\
&\le 2\|K_2\| + \|K_1\|(\|v_n(s)\| + \|v(s)\|
\end{aligned}$$

Since $\lim_{n\to\infty} \|v_n - v\| = 0, \exists \overline{K} > 0$ such that
$\|v_n\| \le \overline{K}, \forall n \in \mathbb{N} \quad and \quad \|v\| \le \overline{K}$. So $s \mapsto \frac{f(s,v_n(s)) - f(s,v(s))}{e^{\alpha s}}$ is bounded. Moreover the continuity of f guarantees that$0 \le s \le T, \lim_{n\to\infty} \|f(s,v_n(s)) - f(s,v(s))\| = 0$ We have that

$$\begin{aligned}
\|(\phi v_n)(t) - (\phi v)(t)\| &\le \int_0^t \|e^{(t-s)A}\|\|f(s,v_n(s)) - f(s,v(s))\|ds \\
&\le M \int_0^t e^{(t-s)\alpha}\|f(s,v_n(s)) - f(s,v(s))\|ds
\end{aligned}$$

Taking the supremum over the interval $[0,T]$, we have

$$\begin{aligned}
\|\phi v_n - \phi v\|_\infty &\le Me^{\alpha T} \int_0^T e^{-\alpha s}\|f(s,v_n(s)) - f(s,v(s))\|ds \\
&\le Me^{\alpha T} \int_0^T \|f(s,v_n(s)) - f(s,v(s))\|ds
\end{aligned}$$

The set $\{\|f(s,v_n(s)) - f(s,v(s))\|, n \in \mathbb{N}, s \in [0,T]\}$ is uniformly bounded and so by the Lebesgue Dominated Convergence Theorem there exists $N \in \mathbb{N}$ such that:
$n \ge N \Rightarrow \int_0^T \|f(s,v_n(s)) - f(s,v(s))\|ds < \frac{\epsilon}{Me^{\alpha T}}$. Hence for all $n \ge N, \|\phi v_n - \phi v\| < \epsilon$ as desired and this proves step 1.

Step 2: We show that ϕ is a completely continuous operator.
Let $k > 0$ and define the ball $\mathcal{B} = \{v \in \mathcal{C}([o,T];\mathbf{X}) : \|v\|_{\mathcal{C}} \le k\}$. We shall show that $\phi(\mathcal{B})$ is precompact in $\mathcal{C}([0,T];\mathbf{X})$ using the Arzela-Ascoli theorem.

Claim 1: For every $0 \le t_0 \le T, \{(\phi v)(t_0) : v \in \mathcal{B}\}$ is precompact in $\mathbf{X}$

Proof: For $t_0 = 0, \{(\phi v)(t_0) = u_0 : v \in \mathcal{B}\} = \{u_0\}$ which is compact as a finite set in $\mathbf{X}$. Now let $0 \le t_0 \le T$ be fixed and $0 < \epsilon < t_0$.
Define $\phi_\epsilon : \mathcal{C}([0,T] : \mathbf{X}) \to \mathcal{C}([0,T];\mathbf{X})$ by

$$(\phi_\epsilon v)(t_0) = e^{t_0 A} + e^{\epsilon A} \int_0^{t_0-\epsilon} e^{(t_0-\epsilon-s)A} f(s,v(s))ds$$

Let $v \in \mathcal{B}$ then we have that

$$\begin{aligned}
\Big\| \int_0^{t_0-\epsilon} e^{(t_0-\epsilon-s)A} f(s,v(s))ds \Big\| &\leq \int_0^{t_0-\epsilon} \|e^{(t_0-\epsilon-s)A}\| \|f(s,v(s))\| ds \\
&\leq \int_0^{t_0-\epsilon} \|e^{(t_0-\epsilon-s)A}\| \big[kK_1(s) + K_2(s)\big] ds \\
&\leq k\int_0^{t_0-\epsilon} \|e^{(t_0-\epsilon-s)A}\| K_1(s) ds + \int_0^{t_0-\epsilon} \|e^{(t_0-\epsilon-s)A}\| K_2(s) ds < \infty
\end{aligned}$$

since $K_1, K_2 \in \mathcal{C}([0,T] : \mathbb{R})$
So the set $\{\int_0^{t_0-\epsilon} e^{(t_0-\epsilon-s)A} f(s,v(s))ds : v \in \mathcal{B}\}$ is bounded. And it therefore follows that the set $K_\epsilon = \{(\phi_\epsilon v)(t_0) : v \in \mathcal{B}\}$ is precompact in $\mathbf{X}$ and hence totally bounded in $\mathbf{X}$.
Next observe that $\forall v \in \mathcal{B}$

$$\begin{aligned}
\|(\phi v)(t_0) - (\phi_\epsilon v)(t_0)\| &\leq \Big\| \int_0^{t_0} e^{(t_0-s)A} f(s,v(s))ds - \int_0^{t_0-\epsilon} e^{(t_0-s)A} f(s,v(s))ds \Big\| \\
&\leq \int_{t_0-\epsilon}^{t_0} \|e^{(t_0-s)A} f(s,v(s))\| ds \\
&\leq Me^{\alpha t_0} \int_{t_0-\epsilon}^{t_0} \|f(s,v(s))\| ds, \quad \alpha \in \mathbb{R} \\
&\leq Me^{\alpha t_0} \int_{t_0-\epsilon}^{t_0} \big[K_1(s)\|v(s)\| + K_2(s)\big] ds \\
&\leq \beta\epsilon
\end{aligned}$$

where $\beta = Me^{\alpha t_0}\big[k\|K_1\| + \|K_2\|\big]$. Thus $\{(\phi v)(t_0) : v \in \mathcal{B}\}$ is totally bounded on $\mathbf{X}$ and hence precompact in $\mathbf{X}$.
Claim2: $\{\phi(v) : v \in \mathcal{B}\}$ is equicontinuous.

Proof: We assume $e^{tA} : [0,\infty) \to \mathbf{X}$ is uniformly continuous. This implies that for all $\epsilon > 0$ there exists $\delta_1 > 0$, such that

$$| t - s | < \delta_1 \Rightarrow \|e^{tA} - e^{sA}\| < \frac{\epsilon}{3(\|u_0\| + 1)} \tag{3.30}$$

$$| t - s | < \delta_1 \Rightarrow \|e^{t-\tau)A} - e^{(s-\tau)A}\| < \frac{\epsilon}{3T(k\|K_1\| + \|K_2\| + 1)} \tag{3.31}$$

Let

$$\delta = \min\left\{\delta_1, \frac{\epsilon}{3Me^{\alpha T}(k\|K_1\| + \|K_2\| + 1)}\right\} > 0 \tag{3.32}$$

Observe that for $0 \leq s < t \leq T$ for which $\mid t-s \mid < \delta$, using (3.30)-(3.32) yields

$$\begin{aligned}
\|(\phi v)(t)-(\phi v)(s)\| &\leq \|(e^{(tA}-e^{sA})u_0\|+\|\int_0^t e^{(t-\tau)A}f(\tau,v(\tau))d\tau-\int_0^s e^{(s-\tau)A}f(\tau,v(\tau))d\tau\| \\
&\leq \|(e^{(tA}-e^{sA})u_0\|+\int_0^s \|e^{(t-\tau)A}-e^{(s-\tau)A}\|\|f(\tau,v(\tau))\|d\tau+ \\
&\quad \int_s^t \|e^{(s-\tau)A}\|\|f(\tau,v(\tau))\|d\tau \\
&\leq \|(e^{(tA}-e^{sA})u_0\|+\int_0^s \|e^{(t-\tau)A}-e^{(s-\tau)A}\|(k\|K_1\|+\|K_2\|)d\tau+ \\
&\quad Me^{\alpha T}(k\|K_1\|+\|K_2\|)\mid t-s\mid \\
&< \epsilon,\quad \forall v\in\mathcal{B}.
\end{aligned}$$

That is $\|(\phi v)(t)-(\phi v)(s)\| < \epsilon \quad whenever \quad \mid t-s \mid < \delta, \forall v \in \mathcal{B}$
This proves the claim and so by Arzela-Ascoli's theorem we conclude that $\phi(\mathcal{B})$ is precompact in $\mathcal{C}([0,T]:\mathbb{R})$.
Thus ϕ is a compact map and this concludes *step*2.

Step 3: We show that the set defined by

$$\xi(\phi)=\{v\in\mathcal{C}([0,T]:\mathbf{X})\,|\,\exists\lambda\geq 1 \quad such \quad that \quad \lambda x=\phi(x)\}$$

is bounded.
Proof: Let $v \in \xi(\phi)$ be fixed. We must find a constant C^* (independent of v and λ) such that $\|v\|_{\mathcal{C}} \leq C^*$. To this end we observe that

$$\begin{aligned}
\|v(t)\| &\leq \lambda\|v(t)\|=\|\lambda v(t)\|=\|(\phi v)(t)\| \\
&\leq \|e^{tA}u_0\|+\int_0^t \|e^{(t-s)A}\|\|f(s,v(s)\|ds \\
&\leq Me^{\alpha T}\|u_0\|+Me^{\alpha T}\int_0^t [K_1(s)\|v(s)\|+K_2(s)]ds \\
&\leq Me^{\alpha T}(\|u_0\|+T\|K_2\|)+\int_0^t (Me^{\alpha T}K_1(s))\|v(s)\|ds
\end{aligned}$$

Let $C_1 = Me^{\alpha T}(\|u_0\| + T\|K_2\|)$. Applying Gronwall's lemma yields

$$\|v(t)\| \leq C_1 e^{\int_0^t Me^{\alpha t}K_1(s)ds} \leq C_1 e^{Me^{\alpha T}\|K_1\|} = C^*,\ 0\leq t\leq T$$

So taking the supremum over $[0,T]$ of the above expression enables us to conclude that

$$\|v\| \leq C^*, \forall v \in \xi(\phi)$$

This completes *step*3. So we have prove that ϕ is compact, continuous and $\xi(\phi)$ is bounded. Thus by virtue of the Schaefer fixed-point theorem, we conclude that ϕ has at least one fixed-point, which coincides with a mild solution of (3.1). Therefore this completes the proof of the theorem.

CHAPTER 4

APPLICATIONS

In this chapter we apply the theory developed in the previous chapters to some linear and non-linear diffusion problems.

4.1 The Homogeneous Heat Equation

Let Ω be an open subset of $\mathbb{R}^N$, with boundary Γ. We denote by
$Q = \Omega \times (0, \infty)$, $\quad \Sigma = \Gamma \times (0, \infty)$. Consider the following problem.

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u &= 0, & \text{in} \quad Q \\ u &= 0, & \text{on} \quad \Sigma \\ u(x, 0) &= u_0(x), & \text{on} \quad \Omega \end{cases} \tag{4.1}$$

where Δ denotes the Laplacian with respect to the space variable, t is the time variable and $u_0(x)$ is a given function. Equation (4.1) is called the Heat equation because it models the distibution of the temperature u in Ω at time t.

We need to reformulate (4.1) into an equation of the form (3.1) and show the existence of a unique mild solution.
Let $u = U$ where U is defined on $[0, \infty)$ with values in the Hilbert space $H = \mathbb{L}^2$. Considering the boundary conditions in (4.1), we defined the unbounded linear operator
$A : \mathcal{D}(A) \subset H \longrightarrow H$ by:

$$\begin{cases} \mathcal{D}(A) &= H^2(\Omega) \cap H_0^1(\Omega) \\ Au &= \Delta u \end{cases}$$

Therefore, we have the follwing abstract Cauchy problem which is a reformulation of (4.1)

$$\begin{cases} \frac{dU}{dt} &= AU, & t > 0 \\ U(0) &= U_0, \end{cases} \tag{4.2}$$

We show that the unbounded operator A is Maximal dissipative.
Dissipative
Let $v \in \mathcal{D}(A)$, we have

$$\begin{aligned} \langle Av, v\rangle_{\mathbb{L}^2} &= \int_\Omega (\Delta v)v dx \\ &= -\int_\Omega (\nabla v)(\nabla v)dx + \int_{\partial\Omega} (\frac{\partial v}{\partial \overrightarrow{n}}.v)d\sigma \\ &\quad \text{by the integration by part} \\ &\leq 0 \end{aligned}$$

Thus $\langle Av, v\rangle \leq 0$ showing dissipativity.
Maximal-dissipative
But from Example 1.2.13, we have that $\forall f \in \mathbb{L}^2$, $\lambda > 0 \quad \exists!\, u \in H^2 \cap H_0^1$ solution of the equation

$$\lambda u - \Delta u = f$$

Thus A is m-dissipative (by Theorem 4.4.2) and therefore generates a $\mathcal{C}_0$-semigroup of contractions $\{e^{tA}, t \geq 0\}$. Therfore (4.2) has a unique mild solution.

4.2 The Classical Wave Equation

The evolution over time of the vertical displacement of some vibrating strings subject to small vibrations are described by the so-called *wave equations*. Suppose that the deflection of the string at position x along the string at time t is given by $z(x,t)$. Arguments based on elementary physical principles yields the following initial boundary value problem (IBVP). Let Ω be bounded regular subset of $\mathbb{R}^N$, with a boundary Γ of class $\mathcal{C}^2$.

$$\begin{cases} \frac{\partial^2 z}{\partial t^2} - \Delta z = 0, & \text{in} \quad Q = \Omega \times (0,T) \\ z(x,0) = z_0(x), \frac{\partial z}{\partial t}(x,0) = z_1(x), & \text{in} \quad \Omega \\ z = 0, & \text{on} \quad \Sigma = \Gamma \times (0,T) \end{cases} \tag{4.3}$$

We formulate (4.3) into an abstract Linear evolution equation, of type (3.1), (with $f \equiv 0$). By suitably choosing the state space X and the operator A. We make the following changes of variables.
$v_1 = z, v_2 = \frac{\partial z}{\partial t} \Rightarrow \frac{\partial v_1}{\partial t} = v_2, \frac{\partial v_2}{\partial t} = \frac{\partial^2 v_1}{\partial t^2}$.
Then we have the following system

$$\begin{cases} u'(t) = Au(t), \quad t > 0 \\ u(0) = u_0 \end{cases} \tag{4.4}$$

where $u = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$$Au = \begin{bmatrix} 0 & I \\ \Delta & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_2 \\ \Delta v_1 \end{bmatrix}$$

$$\frac{d}{dt}u = \frac{\partial}{\partial t}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$u_0 = \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} (x, 0)$

Now we need to construct the state space X and it has to be a product space say $K_1 \times K_2$, since the unknown is a vector consisting of two components. A reasonable choice of K_2 is $\mathbb{L}^2(\Omega)$. For K_1, the first component vanishes at the boundary and is at least one time differentiable. And it turns out that the space $H_0^1(\Omega)$ equiped with the norm
$\|z\|_{H_0^1}^2 = \|\frac{dz}{dx}\|_{\mathbb{L}^2}^2 + \|z\|_{\mathbb{L}^2}^2$ is a Hilbert space. So let's choose $K_1 = H_0^1(\Omega)$. Hence

$$X = H_0^1(\Omega) \times \mathbf{L}^2(\Omega) \tag{4.5}$$

$\left\langle \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \begin{bmatrix} v_1^* \\ v_2^* \end{bmatrix} \right\rangle_X = \int_0^L \left[\frac{\partial v_1}{\partial x} \frac{\partial v_1^*}{\partial x} + v_2 v_2^*\right] dx$

Now define the operator A by $A : \mathcal{D}(A) \subset X \longrightarrow X$

$$Au = \begin{bmatrix} 0 & I \\ \Delta & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_2 \\ \Delta v_1 \end{bmatrix} \tag{4.6}$$

$\mathcal{D}(A) = \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times H_0^1(\Omega)$

Let us show that A is m-dissipative on X, when X is equiped with the inner product

$$\langle u, v \rangle_X = \int_\Omega \nabla u_1 \nabla v_1 + \int_\Omega u_2 v_2, \quad where \quad u = (u_1, u_2) \quad and \quad v = (v_1, v_2)$$

dissipative:

$$\begin{aligned} \langle Av, v \rangle_X &= \int_\Omega \nabla v_1 \nabla v_2 + \int_\Omega \Delta v_1 v_2 \\ &= \int_\Omega \nabla v_1 \nabla v_2 - \int_\Omega \nabla v_1 \nabla v_2 \\ &= 0 \end{aligned}$$

That is $\langle Av, v \rangle_X = 0, \forall v \in X$. And hence A is dissipative. In fact conservative.

m-dissipative

Let $(f, g) \in H_0^1(\Omega) \times \mathbb{L}^2(\Omega)$ and $\lambda > 0$. The equation

$$\lambda u - Au = (f, g)$$

is equivalent to the system

$$\begin{cases} \lambda u_1 - u_2 &= f \\ \lambda u_2 - \Delta u_1 &= g \end{cases}$$

Substituting $u_2 = \lambda u_1 - f$ into the second equation of the system , we get

$$\lambda^2 u_1 - \Delta u_1 = \lambda f + g$$

This equation admits a unique solution $u_1 \in H^2(\Omega) \cap H_0^1(\Omega)$, since the Laplace operator is m-dissipative (by Theorem 4.4.2). Consequently $u_2 \in H_0^1(\Omega)$ is unique. Thus A is m-dissipative. Therefore by Lumer-Phillips Theorem $(A, \mathcal{D}(A))$ is the generator of a $\mathcal{C}_0$-semigroup of contraction on X denoted by $\{e^{tA}, t \geq 0\}$. Further $u_0 = \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} \in X$. Thus (4.4) has a unique mild solution given by $\begin{bmatrix} z(t) \\ \frac{\partial z(t)}{\partial t} \end{bmatrix} = e^{tA} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix}$, which is also a classical solution whenever $\begin{bmatrix} z_0 \\ z_1 \end{bmatrix} \in \mathcal{D}(A)$

4.3 The Nonlinear Heat Equation

Consider the initial boundary value problem for the semilinear Heat Equation:

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u &= f(u), & \text{in} \quad \Omega \\ u &= 0, & \text{on} \quad \Gamma \\ u(x,0) &= u_0(x), & \text{on} \quad \Omega \end{cases} \tag{4.7}$$

where Ω is assumed to be a bounded domain in $\mathbb{R}^n$ with smooth boundary Γ.

Theorem 4.3.1 [9] *Suppose $f(u) = -u^3 + u$, $n \leq 3$ and $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$. Then problem (4.7) admits a unique global classical solution u such that*

$$u \in \mathcal{C}([0,+\infty), H^2 \cap H_0^1) \cap \mathcal{C}^1([0,+\infty), L^2)$$

4.4 Nonlinear Wave Equations

Let $\emptyset \neq \Omega \subset \mathbb{R}^3$ be open and bounded with $\mathcal{C}^2$ boundary. For $a \in \mathbb{R}$, $u_0 \in Y := W_0^{1,2}(\Omega) = H_0^1(\Omega)$ and $u_1 \in \mathbb{L}^2(\Omega)$, we consider the problem.

$$\begin{cases} \partial_{tt} z &= \Delta z - az\|z\|^2_{H_0^1}, & \text{in} \quad \Omega, \ t > 0 \\ z &= 0, & \text{on} \quad \partial\Omega, \ t > 0 \\ z(x,0) &= u_0, & \partial_t z(x,0) = u_1 \end{cases} \tag{4.8}$$

We reformulate (4.8) as an abstract semi-linear Evolution Equation on $X = H_0^1(\Omega) \times \mathbb{L}^2(\Omega)$, using Δu with $\mathcal{D}(\Delta u) = H^2(\Omega) \times H_0^1(\Omega)$.
$A = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}$ with $\mathcal{D}(A) = \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times H_0^1(\Omega)$.
$F(u,v) = (0, -au\|u\|^2_{H_0^1}) =: (0, F_0(u)), \quad$ *for* $\quad (u,v) \in X$.
Let $u = z, v = \frac{\partial z}{\partial t}$ *then* $\frac{\partial v}{\partial t} = \frac{\partial^2 u}{\partial t^2}$. Then we can rewrite equation (4.8) as follows.

$$\begin{cases} \frac{\partial}{\partial t}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}\begin{bmatrix} u \\ v \end{bmatrix} + \begin{pmatrix} 0 \\ -a\|u\|^2_{H_0^1} \end{pmatrix}, \quad on \quad \Omega, \ t > 0 \\ \begin{bmatrix} u \\ v \end{bmatrix}(t) = 0, \qquad on \quad \partial\Omega, \ t > 0 \\ \begin{bmatrix} u \\ v \end{bmatrix}(0) = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \end{cases} \tag{4.9}$$

So we have the following abstract Cauchy problem, for $U = (u,v)$

$$\begin{cases} U'(t) &= AU(t) + F(U(t)), \quad t > 0 \\ U(0) &= U_0 \end{cases} \tag{4.10}$$

where $U_0 = (u_0, u_1)$ and

$$\begin{array}{llll} F : [0,\infty) \times X & \longrightarrow X & & \\ \quad (t,U) & \mapsto & F(t,U) & = F(U(t)) \\ & & & = \Big(0, -aU(t)\|U(t)\|^2_{H^1_0}\Big) \\ & & & = (0, F_0(U)) \end{array}$$

In the sequel we show that F satisfies a Lipschitz-type condition. Let $u, w \in H^1_0(\Omega)$ with $\|u\|_{H^1}, \|w\|_{H^1} \le r$ with $r > 0$. We have that

$$\begin{aligned} \big\|u\|u\|^2 - w\|w\|^2\big\| &= \big\|u\|u\|^2 - w\|u\|^2 + w\|u\|^2 - w\|w\|^2\big\| \\ &= \big\|(u-w)\|u\|^2 + (\|u\|^2 - \|w\|^2)w\big\| \\ &\le \big\|u-w\big\|\|u\|^2 + \|w\|\big\|\|u\|^2 - \|w\|^2\big\| \\ &= \big\|u-w\big\|\|u\|^2 + \|w\|\big\|(\|u\| - \|w\|)(\|u\| + \|w\|)\big\| \\ &\le \big\|u-w\big\|\|u\|^2 + \|u-w\|\big(\|u\|\|w\| + \|w\|^2\big) \\ &= \|u-w\|\big(\|u\|^2 + \|u\|\|w\| + \|w\|^2\big) \\ &\le \frac{3}{2}\big(\|u\|^2 + \|w\|^2\big)\|u-w\|, \quad \textit{since} \quad \|u\|\|w\| \le \frac{1}{2}\big(\|u\|^2 + \|w\|^2\big) \end{aligned}$$

And it follows that

$$\begin{aligned} \big\|u\|u\|^2 - w\|w\|^2\big\|_{\mathbb{L}^2} &\le \frac{3}{2}\Big\|\big(\|u\|^2 + \|w\|^2\big)\|u-w\|\Big\|_{\mathbb{L}^2} \\ &\le C(\Omega)\frac{3}{2}\Big(\|u\|^2_{\mathbb{L}^6} + \|w\|^2_{\mathbb{L}^6}\Big)\|u-w\|_{\mathbb{L}^6} \end{aligned}$$

$C(\Omega)$ is a constant depending on Ω, since $\|f\|_{\mathbb{L}^2} \le [mes(\Omega)]^{\frac{1}{3}}\|f\|_{\mathbb{L}^6}$
Also by the Sobolev embedding $H^1(\Omega) \hookrightarrow \mathbb{L}^6$, we have that $\|u-w\|_{\mathbb{L}^6} \le K\|u-w\|_{H^1(\Omega)}$, and we therefore have
$\big\|u\|u\|^2 - w\|w\|^2\big\|_{\mathbb{L}^2} \le Cr^2\|u-w\|_{H^1(\Omega)}$
Hence $F_0 : H^1_0(\Omega) \longrightarrow \mathbb{L}^2$ is locally Lipschitz and therefore (4.10) has a unique local mild solution U.
Now suppose that $(u_0, u_1) \in \mathcal{D}(A)$. We show that the mild solution U obtained is also a classical solution.
To do this we show that F is continuously Fréchet differentiable.
Let $f(u) = u\|u\|^2 = u\langle u, u\rangle$

$$\begin{aligned} f(u+h) &= (u+h)\langle u+h, u+h\rangle \\ &= (u+h)\big(\langle u,u\rangle + 2\langle u,h\rangle + \langle h,h\rangle\big) \\ &= \big(\|u\|^2 + 2\langle u,h\rangle + \|h\|^2\big)(u+h) \\ &= \|u\|^2u + \|u\|^2h + 2\langle u,h\rangle u + 2\langle u,h\rangle h + \|h\|^2u + \|h\|^2h \\ &= f(u) + \big(\|u\|^2h + 2\langle u,h\rangle u\big) + \big(2\langle u,h\rangle h + \|h\|^2u + \|h\|^2h\big) \end{aligned}$$

But $\big(2\langle u,h\rangle h + \|h\|^2u + \|h\|^2h\big) \longrightarrow 0 \quad as \quad \|h\| \longrightarrow 0$. Thus it follows that
$f'(u)(h) = \|u\|^2h + 2\langle u,h\rangle u$

Now let $u_1, u_2, h \in H_0^2(\Omega)$, Then

$$\begin{aligned}
\big(f'(u_2) - f'(u_1)\big)(h) &= \big(\|u_2\|^2 h + 2\,\langle u_2, h\rangle\, u_2\big) - \big(\|u_1\|^2 h + 2\,\langle u_1, h\rangle\, u_1\big) \\
&= \big(\|u_2\|^2 - \|u_1\|^2\big)(h) + 2\big(\,\langle u_2, h\rangle\, u_2 - \langle u_1, h\rangle\, u_1\big) \\
&= \big(\|u_2\|^2 - \|u_1\|^2\big)(h) + 2\big(\,\langle u_2 - u_1, h\rangle\, u_2 + \langle u_1, h\rangle\, u_2 - \langle u_1, h\rangle\, u_1\big) \\
&= \big(\|u_2\|^2 - \|u_1\|^2\big)(h) + 2\big(\,\langle u_2 - u_1, h\rangle\, u_2 + \langle u_1, h\rangle\, (u_2 - u_1)\big)
\end{aligned}$$

It follows that

$$\begin{aligned}
\|\big(f'(u_2) - f'(u_1)\big)(h)\| &\leq \big|\|u_2\|^2 - \|u_1\|^2\big|\|h\| + 2\|u_2\|\|u_2 - u_1\|\|h\| + 2\|u_1\|\|u_2 - u_1\|\|h\| \\
&= \big|\|u_2\|^2 - \|u_1\|^2\big|\|h\| + 2\big(\|u_2\| + \|u_1\|\big)\|u_2 - u_1\|\|h\|
\end{aligned}$$

It implies that

$$\begin{aligned}
\|f'(u_2) - f'(u_1)\| &\leq \big|\|u_2\|^2 - \|u_1\|^2\big| + 2\big(\|u_2\| + \|u_1\|\big)\|u_2 - u_1\| \\
&\leq 2\big(\|u_1\| + \|u_2\|\big)\|u_2 - u_1\|
\end{aligned}$$

So f' is locally Lipschitz. Thus $f \in \mathcal{C}^1(\Omega)$, and this implies that $F_0 : H_0^1(\Omega) \longrightarrow \mathbb{L}^2$ is is continuously Fréchet differentiable, and as a result F is is continuously Fréchet differentiable on X. Therefore by theorem 3.1.1 the mild solution U of (4.10) is also a classical solution.

Appendix

Let Ω be a bounded open subset of $\mathbb{R}^N$ and consider the problem:

$$\begin{cases} -\Delta u + u &= f, \quad \text{in} \quad \Omega \\ u &= 0, \quad \text{on} \quad \partial\Omega \end{cases} \tag{4.11}$$

Definition 4.4.1 *Let $f \in L^2(\Omega)$. A classical solution of (4.11) is a function $u \in \mathcal{C}^2(\bar{\Omega})$ satisfying (4.11). A function $u \in H_0^1(\Omega)$ is called a weak solution of (4.11) if*

$$\int_\Omega (\nabla u \nabla v)\, dx + \int_\Omega uv dx = \int_\Omega f v dx, \quad \forall v \in H_0^1(\Omega)$$

Lemma 4.4.1 *If $f \in \mathbb{L}^2(\Omega)$ then (4.11) has a unique weak solution $u \in H_0^1(\Omega)$.*
Moreover

$$\|u\|_{H^2} \leq C\|f\|_{\mathbb{L}^2} \tag{4.12}$$

where C is a constant.

To prove the above theorem one would need the following theorem.

Theorem 4.4.1 (Lax-Milgram) *Let H be a Hilbert space and let $a(\cdot,\cdot)$ be a continous and H-elliptic bilinear form on H. Then given $L : H \to \mathbb{R}$ a continuous linear form on H, there exists a unique $u \in H$ such that*

$$a(u, v) = L(v),\, \forall\, u \in H \tag{4.13}$$

Moreover the map $L \mapsto u$ is continuous from H' to H. If in addition the bilinear form a is symmetric, then u is the unique minimizer of the functional
$J(u) = \frac{1}{2}a(a, v) - L(v)$

Proof of Lemma 4.4.1: Multiplying (4.11) by $v \in H_0^1(\Omega)$ and integrating by parts, we get

$$\int_\Omega \nabla u \nabla v dx + \int_\Omega uv dx = \int_\Omega f v dx$$

Now we just apply the Lax-Milgram theorem with

$$H = H_0^1(\Omega),\ a(u,v) = \int_\Omega (\nabla u \nabla v)\, dx + \int_\Omega uvdx \text{ and } L(v) = \int_\Omega fvdx$$

To get (4.12), it suffices to take $v = u$. And it follows that

$$\|u\|_{H^1(\Omega)} \leq \|u\|_{L^2}\|f\|_{L^2} = C\|f\|_{L^2}, \quad \text{where} \quad C = \|u\|_{L^2}$$

Therefore the proof is complete

Proposition 4.4.1 *Assume that Ω is $\mathcal{C}^1$, $f \in \mathcal{C}^0$ and the weak solution $u \in H_0^1(\Omega)$ of (4.11) is in $\mathcal{C}^2(\bar{\Omega})$. Then u is a classical solution.*

Proof. 4.4.2

Since $u \in H_0^1(\Omega)$ and Ω is smooth, we have that $u = 0$ on $\partial\Omega$. On the other hand, u satifies

$$\int_\Omega (\nabla u \nabla \varphi)\, dx + \int_\Omega u\varphi dx = \int_\Omega f\varphi dx, \quad \forall \varphi \in \mathcal{D}(\Omega)$$

it follows that

$$\int_\Omega (-\Delta u + u)\varphi dx = \int_\Omega f\varphi dx, \quad \forall \varphi \in \mathcal{D}(\Omega)$$

So $-\Delta u + u = f$ a.e in Ω. But $-\Delta u + u$ is continuous on Ω, therefore $-\Delta u + u = f$ for all $x \in \Omega$. Thus u is a classical solution of (4.11).

Theorem 4.4.2 *The Laplace operator is m-dissipative.*

Proof. 4.4.3

The proof follows directly from Lemma 4.4.1 and Proposition 4.4.1.

Conclusion and perspective

In this work, we have presented in detail, the theory of Abstract Evolution Equations using the semigroup approach as an effective method to examine the qualitative behavior of solutions to semilinear evolution equations. Also we have applied this theory to some diffusion problems by reformulating them into an abstract Cauchy problem, and made use of the underlying properties.

As perspective, we shall consider other models of equations such as periodic problems, to study the existence of periodic solutions and their qualitative properties. We also hope study other methods (such as the variational methods) and to extend our research to the Semilinear Evolution Inclusions and also to Stochastics Evolution Equations due to our great interest in Probability Theory and Stochastics Analysis.

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