# Differential Forms and Applications

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By

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# Preface

This project deals mainly with Differential Forms on smooth Riemannian manifolds and their applications through the properties of their classical Differential and Integral Operators.

The calculus of Differential Forms provides a simple and flexible alternative to vector calculus. It is not dependent on any coordinate system, simplifies or condenses variational principles, offers a more comprehensive means of evaluating multivariate integrals, and is crucial in the analysis of the variation of differentiable functions on smooth manifolds. Differential Forms have numerous applications within (and beyond) Differential Geometry and Mathematical Physics.

Needless to mention, Differential Forms constitute the ingredients (test functions) of the Theory of $k$-current which is analogous to Distribution Theory, and so they offer diverse potential tools for research.

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# Contents

# Introduction

This body of work introduces exterior calculus in Euclidean spaces and subsequently implements classical results from standard Riemannian geometry to analyze certain differential forms on a manifold of reference, which here is a symmetric ellipsoid in $\mathbb{R}^n$.

We focus on the foundations of the theory of differential forms in a progressive approach to present the relevant classical theorems of Green and Stokes and establish volume (length, area or volume) formulas.

Furthermore, we introduce the notion of geodesics and show how to obtain them with respect to the reference manifold.

# CHAPTER 1

# MANIFOLDS AND FORMS

## 1.1 Submanifolds of $\mathbb{R}^n$ without boundary

**Definition 1.1.1.**
*A subset $M$ of $\mathbb{R}^n$ is called a k-dimensional submanifold without boundary if for each point $p \in M$, there exist $U, V$ open in $\mathbb{R}^n$ with $p \in U$ as well as a diffeomorphism $\phi : U \to V$ such that $\phi(U \cap M)$ is contained in the subspace $\mathbb{R}^k \subseteq \mathbb{R}^n$. In other words,*

$$\phi(U \cap M) = V \cap (\mathbb{R}^k \times \{0\}^{n-k}) = \{y \in V \ : y^{k+1} = \cdots = y^n = 0\}.$$

*The pair $(U^*, \phi)$ where $U^* = U \cap M$ is called a local chart around $p$ and a family of local charts covering all points of M is called an atlas on M. Thus if $\{U_i, \phi_i\}_{i \in I \subseteq \boldsymbol{N}}$ is an atlas on M, then $M = \bigcup_{i \in I} U_i$.*

*Remark:* Because the dimension of M is k, we say that M has a local $\mathbb{R}^k$ property and use this property to create parametrizations for the manifold, which are basically differentiable functions mapping from a subset of $\mathbb{R}^k$ onto M. Parametrizations are needed for computational and analytical purposes as we will see in chapter 2 on examples of differential forms on Riemannian manifolds.

If $(U^*, \phi)$ is a local chart with $p \in U^*$, we can identify $p$ and the vector $\phi(p) \in \mathbb{R}^n$. The coordinates of $\phi(p)$ in $\mathbb{R}^n$ are called the local coordinates of $p$ in the local chart $(U^*, \phi)$. For any two charts $(U_i, \phi_i)$ and $(U_j, \phi_j)$ such that $U_i \cap U_j$ is non-empty, we can define the map,

$$\phi_i \circ {\phi_j}^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$$

which is called a chart transition from one chart to another. The sets $\phi_j(U_i \cap U_j)$ and $\phi_i(U_i \cap U_j)$ are open sets of the coordinate space $\mathbb{R}^k$ and and the transition function $\phi_i \circ {\phi_j}^{-1}$ is a diffeomorphism.
Alternatively, we may define a submanifold of $\mathbb{R}^n$ without boundary as follows.

**Definition 1.1.2.**
*Let $U \subseteq \mathbb{R}^n$ be an open subset and let $f : U \to \mathbb{R}^{n-k}$ be a smooth map. Consider the set $M = \{x \in U : f(x) = 0\}$.*
*If the gradient $Df(x)$ has maximal rank (n-k) at each point $x \in M$, then $M$ is a smooth k-dimensional submanifold of $\mathbb{R}^n$ without boundary.*

*Remark*: This latter definition is derived from the former as a direct application of the implicit function theorem, as we now briefly explain. For an arbitrary point $\xi = (\xi_1, \cdots, \xi_n) \in M \subset U$, we have by the implicit function theorem an open neighborhood A of $(\xi_1, \cdots, \xi_k)$ in $\mathbb{R}^k$ and a smooth function $g : A \to \mathbb{R}^{n-k}$ such that $g(\xi_1, \cdots, \xi_k) = (\xi_{k+1}, \cdots, \xi_n)$ and $f(\rho, g(\rho)) = 0$ for all $\rho \in A$.

Hence, there exists an open neighborhood $U_\xi$ of $\xi$ in U so that

$$U_\xi \cap M = \{x \in U_\xi : g(x_1, \cdots, x_k) = (x_{k+1}, \cdots, x_n)\}.$$

Consider also the smooth function $\bar{g}$ given by

$$\bar{g} : A \times \mathbb{R}^{n-k} \supset U_\xi \to \mathbb{R}^{n-k}; x \mapsto g(x_1, \cdots, x_k)$$

which belongs to the same diffeomorphism class as $f|_{U_\xi}$ so that there also exists a diffeomorphism $\phi_\xi : U_\xi \to \phi_\xi(U_\xi) \subset \mathbb{R}^n$ such that the map $f \circ \phi_\xi^{-1} : \phi_\xi(U_\xi) \to \mathbb{R}^{n-k}$ is given by the formula

$$f \circ \phi_\xi^{-1}(x_1, \cdots, x_n) = (x_{k+1}, \cdots, x_n)$$

which implies that

$$f(U_\xi \cap M) = f \circ \phi_\xi^{-1}(\phi_\xi(U_\xi) \cap (\mathbb{R}^k \times \{0\}^{n-k})) = 0,$$

i.e.

$$U_\xi \cap M = \phi_\xi^{-1}(\phi_\xi(U_\xi) \cap (\mathbb{R}^k \times \{0\}^{n-k}))$$

since there are no other points in $U_\xi$ whose image under $f$ is 0.

This means $\phi_\xi(U_\xi \cap M) = \phi_\xi(U_\xi) \cap (\mathbb{R}^k \times \{0\}^{n-k})$ and the local chart $(U_\xi \cap M, \phi_\xi)$ needed around $\xi$ is thereby obtained, making M a smooth k-dimensional submanifold of $\mathbb{R}^n$ without boundary.

Let us consider the generalized case in $\mathbb{R}^{n+1}$ of an ellipsoid with axial symmetry.

**Definition 1.1.3.**

*Define*
$M_{(n)} := \left\{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} : \frac{x_1^2}{a^2} + \frac{x_2^2}{a^2} + \cdots + \frac{x_n^2}{a^2} + \frac{x_{n+1}^2}{b^2} = 1\right\}$;
$a, b \in \mathbb{R}^+\backslash\{0\}$. *$M_{(n)}$ is the generalized ellipsoid in $\mathbb{R}^{n+1}$ with the $x_{n+1}$ axis of symmetry.*

$M_{(n)}$ is a differentiable submanifold of $\mathbb{R}^{n+1}$ without boundary, having dimension n. We may justify this statement using the latter description of submanifolds without boundary given in definition 1.1.2.

Consider the function

$$f : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$$

defined by

$$x = (x_1, \cdots, x_{n+1}) \longmapsto f(x) = \|bx\|^2 - a^2b^2 + (a^2 - b^2)x_{n+1}^2$$

$$M_{(n)} = \{x \in \mathbb{R}^{n+1} : f(x) = 0\} \text{ and}$$

$$Df(x) = (2b^2x_1, 2b^2x_2, \cdots, 2b^2x_n, 2a^2x_{n+1}).$$

The rank of the $1 \times (n+1)$ matrix $Df(x)$ is strictly 1 because the $x_i$'s cannot be simultaneously zero since the real constants $a$ and $b$ are positive. This means that $M_{(n)}$ is a manifold of dimension $(n+1) - 1 = n$.

An atlas $\{(U_1, \phi_1), (U_2, \phi_2)\}$ for $M_{(n)}$ is given as follows.

$U_1 = M_{(n)} \backslash \{(0,0,\cdots,0,b)\}$ and

$$\phi_1 : U_1 \longrightarrow \mathbb{R}^n$$

$$x = (x_1, \cdots, x_{n+1}) \longmapsto \phi_1(x) = \left( \frac{\frac{x_1}{a}}{1 - \frac{x_{n+1}}{b}}, \frac{\frac{x_2}{a}}{1 - \frac{x_{n+1}}{b}}, \cdots, \frac{\frac{x_n}{a}}{1 - \frac{x_{n+1}}{b}} \right).$$

$U_2 = M_{(n)} \backslash \{(0,0,\cdots,0,-b)\}$ and

$$\phi_2 : U_2 \longrightarrow \mathbb{R}^n$$

$$x = (x_1, \cdots, x_{n+1}) \longmapsto \phi_2(x) = \left( \frac{\frac{x_1}{a}}{1 + \frac{x_{n+1}}{b}}, \frac{\frac{x_2}{a}}{1 + \frac{x_{n+1}}{b}}, \cdots, \frac{\frac{x_n}{a}}{1 + \frac{x_{n+1}}{b}} \right).$$

These charts are obtained as compositions of stereographic projections ($h_1$ and $h_2$) of the unit sphere $S^n := \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ onto $\mathbb{R}^n$ with the linear map; $T : M_{(n)} \to S^n$ given by $(x_1, \cdots, x_{n+1}) \mapsto \left(\frac{x_1}{a}, \frac{x_2}{a}, \cdots, \frac{x_n}{a}, \frac{x_{n+1}}{b}\right)$.

Indeed, we have
$h_1 : S^n - \{(0,0,\cdots,0,1)\} \to \mathbb{R}^n$,
with $h_1(x_1, \cdots, x_{n+1}) = \left( \frac{x_1}{1 - x_{n+1}}, \cdots, \frac{x_n}{1 - x_{n+1}} \right)$,
$h_2 : S^n - \{(0,0,\cdots,0,-1)\} \to \mathbb{R}^n$,
with $h_2(x_1, \cdots, x_{n+1}) = \left( \frac{x_1}{1 + x_{n+1}}, \cdots, \frac{x_n}{1 + x_{n+1}} \right)$,

so that $\phi_1 = h_1 \circ T$ and $\phi_2 = h_2 \circ T$. Since $h_1$ and $h_2$ are surjective, the maps $\phi_1$ and $\phi_2$ are onto $\mathbb{R}^n$.

Parametrizations of the manifold are considered in the second chapter. In the remainder of this section, we give important properties of $M_{(n)}$ in connection with its submanifolds.

**Definition 1.1.4.**
*Let $M$ be a differentiable manifold of dimension $n$. A submanifold of dimension $d \leq n$ of $M$ is a subset $W \subset M$ such that for any point $p \in W$, there exists a local chart $(\Omega, \phi)$ around $p$ such that $\phi(\Omega \cap W) = U \times V$,*
*$U \subset \mathbb{R}^d, V \subset \mathbb{R}^{n-d}$ and $\phi(\Omega \cap W) = U \times \{0\}^{n-d}$. Thus, there exists a system*

*of local coordinates* $(x_1, \cdots, x_n)$ *on* $\Omega$ *in which the submanifold* $W$ *is locally defined by the equations:* $x_{d+1} = x_{d+2} = \cdots = x_n = 0$.

**Proposition 1.1.5.**
$M_{(d)}$ *is isometrically isomorphic to a submanifold of* $M_{(n)}$ *for* $d \leq n$.

*Proof.* Take an arbitrary point $p \in M_{(d)} \subseteq \mathbb{R}^{d+1}; p = (x_1, \cdots, x_{d+1})$. Clearly, $\mathbb{R}^{d+1}$ is isometrically isomorphic to $\{0\}^{n-d} \times \mathbb{R}^{d+1} \subset \mathbb{R}^{n+1}$, where $\{0\}^{n-d}$ is the zero vector in $\mathbb{R}^{n-d}$. We label the associated isomorphism $I$ and note it acts as follows.

$$I : M_{(d)} \longrightarrow M_{(n)}, \quad p \longmapsto I(p) := p' = (\overbrace{0, 0, \cdots, 0}^{n-d}, x_1, \cdots, x_{d+1})$$

Hence, for each point $p \in M_{(d)}$, the local chart around $p' = I(p)$ (which is either $(U_1, \phi_1)$ or $(U_2, \phi_2)$ as specified above) has the following property, $\phi_i(U_i \cap I(M_{(d)})) = \{0\}^{n-d} \times \mathbb{R}^d$. This is to say that $M_{(d)}$ is isometrically isomorphic to $I(M_{(d)})$; a submanifold of $M_{(n)}$. □

By a similar approach, we also see that the sphere $a.S^d$ is a submanifold of $M_{(n)}$ for $d \leq n-1$, where $S^d = \{x \in \mathbb{R}^{d+1} : \|x\| = 1\}$ is the unit sphere in $\mathbb{R}^{d+1}$ .

More precisely, we have that $M_{(d)} \cong M_{(n)} \cap (\{0\}^{n-d} \times \mathbb{R}^{d+1})$ and $a.S^d \cong M_{(n)} \cap (\mathbb{R}^{d+1} \times \{0\}^{n-d})$, adhering to the axial orientation specified in definition 1.1.3.

Cartesian products of manifolds may be defined when appropriate with $dim(A \times B) = dim(A) + dim(B)$ for manifolds A and B. Nevertheless, it is clearer that we can obtain the ellipsoid $M_{(n)}$ as a manifold of revolution. We specify how to obtain $M_{(n)}$ by revolution in the following proposition.

**Proposition 1.1.6.**
*Let* $\boldsymbol{K}^{n+1} = \{x \in \mathbb{R}^{n+1} : x_1 = 0$ *and* $x_2, \cdots, x_n \geq 0\}$. *Then the manifold* $M_{(n)}$ *is recovered by rotating its (n-1) dimensional submanifold* $M_{(n)} \cap \boldsymbol{K}^{n+1}$ *completely about the* $x_{n+1}$ *axis for* $n \geq 2$.

*Proof.* Each point $p \in \mathbb{R}^{n+1}$ can be given a polar coordinate $(r_p, (\theta_1)_p, \cdots, (\theta_n)_p)$ where $r_p$ is the Euclidean distance from p to the origin and $(\theta_k)_p$ is the angular position of p in the $(x_k, x_{n+1})$ plane measured counterclockwise from the $x_k$ axis. Hence for a point $p' \in M_{(n)} \cap \mathbf{K}^{n+1}$, we have $\{(\theta_1)_{p'}, \cdots, (\theta_n)_{p'}\}$ as a subset of $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

Rotation of a manifold about the $x_{n+1}$ axis entails rotation of its cross-sections about the $x_{n+1}$ axis in each $(x_k, x_{n+1})$ plane. As such, by one full revolution about the $x_{n+1}$ axis, the angular positions of the points (mod $2\pi$) are no longer restricted in any $(x_k, x_{n+1})$ plane for $2 \leq k \leq n$. This eliminates the restriction $x_2, \cdots, x_n \geq 0$ from the result of revolving $M_{(n)} \cap \mathbf{K}^{n+1}$ about the $x_{n+1}$ axis; which is a submanifold of $M_{(n)}$ by virtue of its symmetry about the $x_{n+1}$ axis.

Moreover, by revolving an arbitrary point with Euclidean coordinates $(0, x'_2, x'_3, \cdots, x'_{n+1})$ about the $x_{n+1}$ axis, the result is the sphere in $\{x \in \mathbb{R}^{n+1} : x_{n+1} = x'_{n+1}\}$ centered at $(0, 0, \cdots, 0, x'_{n+1})$ with radius $\|(0, x'_2, x'_3, \cdots, x'_n, 0)\|_2$. But the $x_1$ coordinates of this sphere are clearly not restricted to zero as long as at least one of $x'_2, x'_3, \cdots, x'_n$ is not zero. If $(0, x'_2, x'_3, \cdots, x'_{n+1}) \in M_{(n)} \cap \mathbf{K}^{n+1}$, then its sphere by revolution about the axis of symmetry is a submanifold of $M_{(n)}$, meaning that the $x_1$ coordinates of the manifold by revolution about the $x_{n+1}$ axis are no longer restricted to 0. In conclusion, by revolving $M_{(n)} \cap \mathbf{K}^{n+1}$ about the $x_{n+1}$ axis of symmetry, we get a submanifold of $M_{(n)}$ without the restrictions $x_1 = 0$ and $x_2, \cdots, x_n \geq 0$, which is necessarily a recovery of $M_{(n)}$.

□

More generally, if

$$K = \{x \in \mathbb{R}^{n+1} : x_1 = x_2 = \cdots = x_k = 0 \ \ and \ \ x_{k+1}, \cdots, x_n \geq 0\},$$

then by a similar construction we recover $M_{(n)}$ by rotating its (n-k) dimensional submanifold $M_{(n)} \cap K$ completely about the $x_{n+1}$ axis. The simplest case is by rotating the 1-dimensional half ellipse given by

$$\left\{x \in \mathbb{R}^{n+1} : x_1 = x_2 = \cdots = x_{n-1} = 0, x_n \geq 0 \ \text{ and } \ \frac{x_n^2}{a^2} + \frac{x_{n+1}^2}{b^2} = 1\right\}$$

about the $x_{n+1}$ axis in the space $\mathbb{R}^{n+1}$ to get $M_{(n)}$. This geometric property of $M_{(n)}$ will be reconciled to differential forms on the manifold in further analytic and theoretic observations. (See section 1 of chapter 2 on the volume element of $M_{(2)}$.)

## 1.2 Notions of forms and fields

### 1.2.1 Tensors

A mapping $T$ from $V$, an n-dimensional vector space over $\mathbb{R}$, to $\mathbb{R}$ is called a k-tensor on $V$ if $T : V^k \to \mathbb{R}$ is k-linear. In other words, $T$ is a k-tensor on $V$ iff the following two conditions hold:

**i)** $T(v_1, \cdots, v_i+v'_i, \cdots, v_k) = T(v_1, \cdots, v_i, \cdots, v_k)+T(v_1, \cdots, v'_i, \cdots, v_k)$,

**ii)** $T(v_1, \cdots, av_i, \cdots, v_k) = aT(v_1, \cdots, v_i, \cdots, v_k)$.

The set of all k-tensors on $V$ constitutes a co-vector space denoted $\Im^k(V)$. For $S \in \Im^k(V)$ and $T \in \Im^l(V)$, their tensor product $S\otimes T$ belongs to $\Im^{k+l}(V)$ and is defined by
$S \otimes T(v_1, \cdots, v_k, v_{k+1}, \cdots, v_{k+l}) = S(v_1, \cdots, v_k).T(v_{k+1}, \cdots, v_{k+l})$

A k-tensor $T$ is said to be alternating if

$$T(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) = -T(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_k).$$

The subset of alternating k-tensors in $\Im^k(V)$ also constitutes a co-vector space denoted $\bigwedge^k(V)$. For every $T \in \Im^k(V)$,we generally define Alt(T) by

$$Alt(T)(v_1, \cdots, v_k) = \frac{1}{k!}\sum_{\sigma\in S_k} sgn\sigma.T(v_{\sigma(1)}, \cdots, v_{\sigma(k)})$$

where $S_k$ is the set of all permutations of the integers 1 to k.

We observe that $Alt(T) \in \bigwedge^k(V)$. For $\omega \in \bigwedge^k(V), \eta \in \bigwedge^l(V)$, we define their wedge product or exterior product denoted $\omega \wedge \eta$ which belongs to $\bigwedge^{k+l}(V)$ by

$$\omega \wedge \eta = \frac{(k+l)!}{k!l!}Alt(\omega \otimes \eta).$$

We also give the following properties of the wedge product

1. $(\omega + \eta) \wedge \theta = \omega \wedge \theta + \eta \wedge \theta$ $\quad \forall\omega, \eta \in \bigwedge^k(V); \theta \in \bigwedge^l(V)$
2. $\omega \wedge (\eta + \theta) = \omega \wedge \eta + \omega \wedge \theta$ $\quad \forall\omega \in \bigwedge^k(V); \eta, \theta \in \bigwedge^l(V)$
3. $a\omega \wedge \eta = \omega \wedge (a\eta) = a(\omega \wedge \eta)$ $\quad \forall a \in \mathbb{R}, \omega \in \bigwedge^k(V), \eta \in \bigwedge^l(V)$

4. $(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta)$ $\quad \forall \omega \in \bigwedge^k(V), \eta \in \bigwedge^l(V), \theta \in \bigwedge^m(V)$

5. $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$ $\quad \forall \omega \in \bigwedge^k(V), \eta \in \bigwedge^l(V)$

$\Im^1(V)$ is the set of all linear maps from V to $\mathbb{R}$, which in this case coincides with the dual $V^*$ of $V$ because $V$ is finite dimensional. If $(v_1, \cdots, v_n)$ is a basis for V and $(\varphi_1, \cdots, \varphi_n)$ the corresponding dual basis then the set of all k-fold tensor products $\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k} : 1 \leq i_1, \cdots, i_k \leq n$ is a basis for $\Im^k(V)$, hence having dimension $n^k$. Note that $\varphi_i(v_j) = 0$ when $i \neq j$ and $\varphi_i(v_i) = 1$.

The set of all $\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k} : 1 \leq i_1 < i_2 < \cdots < i_k \leq n$ is a basis for $\bigwedge^k(V)$ which therefore has dimension $\begin{pmatrix} n \\ k \end{pmatrix} = \dfrac{n!}{k!(n-k)!}$

For a differentiable function $f : \mathbb{R}^n \to \mathbb{R}, Df(p)_{p \in \mathbb{R}^n}$ is an example of a linear map from $\mathbb{R}^n$ to $\mathbb{R}$ so $Df(p) \in \bigwedge^1(\mathbb{R}^n) = \Im^1(\mathbb{R}^n)$.

An inner product $T : V \times V \to \mathbb{R}$ is a bilinear functional or 2-tensor on V which is symmetric, that is, $T(v, w) = T(w, v)$ $\forall v, w \in V$ and positive definite, that is, $T(v, v) > 0$ if $v \neq 0$. Hence, T $\in \Im^2(V)$ and we recognize $\langle , \rangle$ as the usual inner product on $\mathbb{R}^n$.

A symplectic map $A : V \times V \to \mathbb{R}$ is another type of bilinear functional on $V$ which is anti-symmetric, that is, $A(v, w) = -A(w, v)$ $\forall v, w \in V$ and so satisfies $A(v, v) = 0$ for all $v \in V$. $A$ is also non-degenerate, meaning that $A(u, v) = 0$ for all $v \in V$ if and only if $u = 0$. Hence $A \in \bigwedge^2(V)$ and we specifically identify such an alternating 2-tensor in section 1 of chapter 2.

Let us examine the vector space $\bigwedge^n(V)$ which has dimension $\begin{pmatrix} n \\ n \end{pmatrix} = 1$. Because of this singular dimension, each element of $\bigwedge^n(V)$ is simply a scalar product of any other non-zero one. The determinant is clearly an alternating n-tensor on V and we state this as $det \in \bigwedge^n(V)$. This fact comes to play in the following theorem.

**Theorem 1.2.1.**
*Let $(v_1, \cdots, v_n)$ be a basis for $V$ and let $\omega \in \bigwedge^n(V)$. If $w_i = \sum_{j=1}^n a_{ij} v_j$ are n vectors in V, then $\omega(w_1, \cdots, w_n) = det(a_{ij}).\omega(v_1, \cdots, v_n)$.*

*Proof.* We first define $\eta \in \Im^n(\mathbb{R}^n)$ by

$$\begin{aligned}\eta((a_{11},\cdots,a_{1n}),\cdots,(a_{n1},\cdots,a_{nn})) &= \omega(\textstyle\sum_{j=1}^n a_{1j}v_j,\cdots\sum_{j=1}^n a_{nj}v_j) \\ &= \omega(w_1,\cdots,w_n).\end{aligned}$$

Then $\eta \in \bigwedge^n(\mathbb{R}^n)$ so $\eta = \lambda.det$ for some constant $\lambda \in \mathbb{R}$. Now, applying both sides to $(e_1,\cdots,e_n)$, we get $\lambda = \eta(e_1,\cdots,e_n) = \omega(v_1,\cdots,v_n)$. As such,

$$\eta((a_{11},\cdots,a_{1n}),\cdots,(a_{n1},\cdots,a_{nn})) = \lambda.det(a_{ij})$$

which implies

$$\omega(w_1,\cdots,w_n) = det(a_{ij}).\omega(v_1,\cdots,v_n).$$

□

As a consequence of theorem 1.2.1, a non-zero $\omega \in \bigwedge^n(V)$ splits the bases of V into two disjoint groups, those with $\omega(v_1,\cdots,v_n) > 0$ and those with $\omega(v_1,\cdots,v_n) < 0$. If $(v_1,\cdots,v_n)$ and $(w_1,\cdots,w_n)$ are two bases such that $w_i = \sum_{j=1}^n a_{ij}v_j$, then these two bases are in the same group iff $det(a_{ij}) > 0$. Either of the two disjoint groups is called an orientation for V. The orientation to which a basis $(v_1,\cdots,v_n)$ belongs is denoted $[v_1,\cdots,v_n]$ and the other orientation is denoted $-[v_1,\cdots,v_n]$. Notably, orientations are independent of the element $\omega$ which acts, meaning that $\omega \neq 0$ separates bases into the same orientations. In $\mathbb{R}^n$, the usual orientation is defined as $[e_1,\cdots,e_n]$.

There is a unique $\omega \in \bigwedge^n(V)$ such that $\omega(v_1,\cdots,v_n) = 1$ whenever $v_1,\cdots,v_n$ is an orthonormal basis such that $[v_1,\cdots,v_n] = \mu$. This unique $\omega$ is called the volume element of V determined by the orientation $\mu$ and an inner product. The determinant is the volume element of $\mathbb{R}^n$ determined by the usual inner product and usual orientation.

### 1.2.2 forms and vector fields on $\mathbb{R}^n$

Let $p \in \mathbb{R}^n$. The set of all pairs $(p,v)$ for $v \in \mathbb{R}^n$ is denoted $\mathbb{R}^n_p$ and is called the tangent space of $\mathbb{R}^n$ at p, i.e. $\mathbb{R}^n_p := (p,v),\ v \in \mathbb{R}^n$. This set is clearly made a vector space by defining the following operations :

1. $(p,v)+(p,w) = (p,v+w)$; $v,w \in \mathbb{R}^n$,
2. $a(p,v) = (p,av)$; $a \in \mathbb{R}$.

A vector $v \in \mathbb{R}^n$ can be seen as an arrow from 0 to $v$, and the vector $(p, v) \in \mathbb{R}^n{}_p$ is then seen as an arrow with the same direction and length, but with initial point p. The vector $(p, v)$ then goes from p to $p+v$ and we write $(p, v)$ as $v_p$ and call it the vector $v$ at $p$.

We define the usual inner product $\langle , \rangle_p$ for $\mathbb{R}^n{}_p$ by $\langle v_p, w_p \rangle_p = \langle v, w \rangle_p$ and assign to $\mathbb{R}^n{}_p$ its usual orientation $:= [(e_1)_p, \cdots, (e_n)_p]$.

A *vector field* on $\mathbb{R}^n$ is a function $F : \mathbb{R}^n \to \mathbb{R}^n{}_p$ such that $F(p) \in \mathbb{R}^n{}_p$ for each $p \in \mathbb{R}^n$. Hence any vector field F can be written as $F(p) = \sum_{i=1}^n F^i(p).(e_i)_p$ thereby yielding n component functions $F^i : \mathbb{R}^n \to \mathbb{R}$.
A similar structure can be placed only on open subsets of $\mathbb{R}^n$. For an open subset U of $\mathbb{R}^n$, we define a vector field as a function which assigns to each point $p \in U$ a unique vector from the tangent space of $\mathbb{R}^n$ at $p$.

The divergence of $F$ ($divF$) is defined as $\sum_{i=1}^n D_i F^i$. Employing the notation $\nabla = \sum_{i=1}^n D_i.e^i$, we may symbolize div $F$ by $\langle \nabla, F \rangle$. Note that $D_i = \dfrac{\partial}{\partial x_i}$.
For n = 3, we can define a vector field called the curl of $F$ or curl $F$, which we symbolize $\nabla \times F$ in accordance with the notation for $\nabla$. Hence,
$(\nabla \times F)(p) = (D_2F^3 - D_3F^2)(e_1)_p + (D_3F^1 - D_1F^3)(e_2)_p + (D_1F^2 - D_2F^1)(e_3)_p$.

Now, a function $\omega$ with $\omega(p) \in \bigwedge^k(\mathbb{R}^n{}_p)$ is called a **k-form on** $\mathbb{R}^n$ or a differential form. If $\varphi_1(p), \cdots, \varphi_n(p)$ is the dual basis to $(e_1)_p, \cdots, (e_n)_p$ then $\omega(p)$ is of the appearance

$$\sum_{i_1 < \cdots < i_k} \omega_{i_1} \cdots \omega_{i_k} \varphi_{i_1}(p) \wedge \cdots \wedge \varphi_{i_k}(p)$$

for certain functions or 0-forms $\omega_{i_1, \cdots, i_k}$. Functions written as $f$ which map to $\mathbb{R}$ are 0-forms. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable so that $Df(p) \in \bigwedge^1(\mathbb{R}^n)$. We then obtain an associated 1-form df, defined by $df(p)(v_p) = Df(p)(v)$.

Upon consideration of the projection maps $\pi_i$ otherwise denoted $dx_i$, for $(1 \leq i \leq n)$, we observe that these belong to the dual of $\mathbb{R}^n$ and
$dx_i(p)(e_i)_p = 1$ ; $dx_i(p)(e_j)_p = 0$ whenever $i \neq j$.
This immediately gives us that $dx_1(p), \cdots, dx_n(p)$ is the dual basis to $(e_1)_p, \cdots, (e_n)_p$, so if $\omega(p)$ is a k-form on $\mathbb{R}^n{}_p$ it can always be written as

$$\sum_{i_1<\cdots<i_k} \omega_{i_1}\cdots\omega_{i_k}(p)dx_{i_1}(p)\wedge\cdots\wedge dx_{i_k}(p).$$

For a differentiable map $f:\mathbb{R}^n\to\mathbb{R}$, $df = D_1f.dx_1+\cdots+D_nf.dx_n$.

For $f:\mathbb{R}^n\to\mathbb{R}^m$ differentiable, we have a linear map $Df(p):\mathbb{R}^n\to\mathbb{R}^m$ to which is associated the linear transformation $f_*:\mathbb{R}^n{}_p\to\mathbb{R}^m{}_{f(p)}$ defined by $f_*(v_p)=(Df(p)(v))_{f(p)}$.

The above induces another linear transformation called the pullback of $f$, written $f^*:\bigwedge^k(\mathbb{R}^m{}_{f(p)})\to\bigwedge^k(\mathbb{R}^n{}_p)$. Therefore if $\omega$ is a k-form on $\mathbb{R}^m$, we define a k-form $f^*\omega$ on $\mathbb{R}^n$ by $(f^*\omega)(p)=f^*(\omega(f(p)))$. This simply means that if $v_1,\cdots,v_k\in\mathbb{R}^n{}_p$, then we have $f^*\omega(p)(v_1,\cdots,v_k)=\omega(f(p))(f_*(v_1),\cdots,f_*(v_k))$. Let $\omega$ be a k-form, then the differential operator $(d)$ acts on $\omega$ to produce a (k+1)-form $d\omega$ which is called the differential of $\omega$.

In general, if

$$\omega=\sum_{i_1<\cdots<i_k}\omega_{i_1}\cdots\omega_{i_k}dx_{i_1}\wedge\cdots\wedge dx_{i_k},$$

then

$$\begin{aligned} d\omega &= \sum_{i_1<\cdots<i_k} d(\omega_{i_1}\cdots\omega_{i_k})\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_k} \\ &= \sum_{i_1<\cdots<i_k}\sum_{\alpha=1}^{n} D_\alpha(\omega_{i_1}\cdots\omega_{i_k})dx_\alpha\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_k} \end{aligned}$$

*Definitions*
The *hodge* operator, denoted $*$, is a linear operator on $\bigwedge^k(\mathbb{R}^n{}_p)$ which assigns an (n-k)-form to each k-form. It has the following property which describes it concisely;

$$*(dx_{i_1}(p)\wedge\cdots\wedge dx_{i_k}(p)) = dx_{i_{k+1}}(p)\wedge\cdots\wedge dx_{i_n}(p),$$

where $(i_1,\cdots,i_k,i_{k+1},\cdots,i_n)$ is an even permutation of the integers from 1 to n.
For $\omega\in\bigwedge^k(\mathbb{R}^n{}_p)$, the (n-k)-form $*\omega$ is called the hodge dual of $\omega$. Note that

$$**\omega=(-1)^{k(n-k)}\omega.$$

An important application of the *hodge* operator is to define the codifferential ($\delta$) of forms. For a k-form $\omega$, we have its codifferential given by

$$\delta\omega = (-1)^{nk+n+1} * d * \omega.$$

Hence, we see that $\delta : \bigwedge^k(\mathbb{R}^n{}_p) \to \bigwedge^{k-1}(\mathbb{R}^n{}_p)$.
The Laplace - Beltrami operator $\Delta : \bigwedge^k(\mathbb{R}^n{}_p) \to \bigwedge^k(\mathbb{R}^n{}_p)$ is given by

$$\Delta = d\delta + \delta d.$$

In separate outstanding considerations, the hodge operator, codifferential and Laplace - Beltrami operator are important tools used in the analysis of Hodge theory.

Important properties of $f^*$; the pullback of $f$ for $f : \mathbb{R}^n \to \mathbb{R}^m$; $(u_1, u_2, \cdots, u_n) \mapsto (x_1, x_2, \cdots, x_m)$ differentiable are listed below

1. $f^*(dx_i) = \sum_{j=1}^n D_j f^i du_j = df^i$
2. $f^*(\omega_1 + \omega_2) = f^*(\omega_1) + f^*(\omega_2)$
3. $f^*(g \cdot \omega) = (g \circ f)f^*\omega$; for a functional $g : \mathbb{R}^m \to \mathbb{R}$
4. $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$
5. If n = m, then $f^*(hdx_1 \wedge \cdots \wedge dx_n) = (h \circ f)(detf')du_1 \wedge \cdots \wedge du_n$
6. $f^*(d\omega) = d(f^*\omega)$

Concerning the differential operator d, there are yet some important observations to make. We have $d^2 = 0$, which is to say $d(d\omega) = 0$ for any differential form $\omega$. Also, $dx_i \wedge dx_i = (-1)^1 dx_i \wedge dx_i = 0$ and $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$ for a k-form $\omega$ and an $l$-form $\eta$.

A form $\omega$ is closed if $d\omega = 0$ and exact if $\omega = d\eta$ for some form $\eta$. Every exact form is closed since if $\omega = d\eta$ then $d\omega = d(d\eta) = 0$. The converse does not necessarily hold. The next theorem gives a sufficient condition for closed forms to be exact.

**Theorem 1.2.2.** ***(Poincaré Lemma)***
*Let $W \subset \mathbb{R}^n$ be an open set star-shaped with respect to the origin, then every closed form on $W$ is exact. A set is said to be star-shaped with respect to the origin if it includes the origin as well as the entire line segment connecting the origin to each of its other points.*

*Proof.* We define a function $I$ from k-forms to (k-1)-forms (for each k), such that $I(0) = 0$ and $\omega = I(d\omega) + d(I\omega)$ for any form $\omega$. It follows that $\omega = d(I\omega)$ if $d\omega = 0$.

Let

$$\omega = \sum_{i_1<\cdots<i_k} \omega_{i_1,\cdots,i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Since A is star-shaped we can define

$$I\omega(x) = \sum_{i_1<\cdots<i_k} \sum_{\alpha=1}^{k} (-1)^{\alpha-1} \left( \int_0^1 t^{k-1} \omega_{i_1,\cdots,i_k}(tx)dt \right) x_{i_\alpha} dx_{i_1} \wedge \cdots \wedge \underline{dx_{i_\alpha}} \wedge \cdots \wedge dx_{i_k}$$

(The strikethrough beneath $dx_{i_\alpha}$ indicates that it is omitted from the term.)

We now prove that $\omega = I(d\omega) + d(I\omega)$.
By Leibnitz's rule,

$$\begin{aligned} d(I\omega) &= k. \sum_{i_1<\cdots<i_k} \left( \int_0^1 t^{k-1} \omega_{i_1,\cdots,i_k}(tx)dt \right) dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &+ \sum_{i_1<\cdots<i_k} \sum_{\alpha=1}^{k} \sum_{j=1}^{n} (-1)^{\alpha-1} \left( \int_0^1 t^k D_j(\omega_{i_1,\cdots,i_k})(tx)dt \right) x_{i_\alpha} \\ & \quad dx_j \wedge dx_{i_1} \wedge \cdots \wedge \underline{dx_{i_\alpha}} \wedge \cdots \wedge dx_{i_k}. \end{aligned}$$

We also have

$$d\omega = \sum_{i_1<\cdots<i_k} \sum_{j=1}^{n} D_j(\omega_{i_1,\cdots,i_k}) dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Applying $I$ to the (k+1)-form $d\omega$, we obtain

$$I(d\omega) = A + B$$

where

$$A = \sum_{i_1<\cdots<i_k} \sum_{j=1}^{n} \left( \int_0^1 t^k D_j(\omega_{i_1,\cdots,i_k})(tx)dt \right) x_j dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

and

$$B = \sum_{i_1<\cdots<i_k} \sum_{\alpha=1}^{k} \sum_{j=1}^{n} (-1)^{\alpha} \left( \int_0^1 t^k D_j(\omega_{i_1,\cdots,i_k})(tx)dt \right) x_{i_\alpha} dx_j \wedge dx_{i_1} \wedge \cdots \wedge \underline{dx_{i_\alpha}} \wedge \cdots \wedge dx_{i_k}$$

Adding, the triple sums cancel, and we obtain

$$\begin{aligned} d(I\omega) + I(d\omega) &= \sum_{i_1<\cdots<i_k} k. \left( \int_0^1 t^{k-1} \omega_{i_1,\cdots,i_k}(tx)dt \right) dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &+ \sum_{i_1<\cdots<i_k} \sum_{j=1}^{n} \left( \int_0^1 t^k x_j D_j(\omega_{i_1,\cdots,i_k})(tx)dt \right) dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &= \sum_{i_1<\cdots<i_k} \left( \int_0^1 \frac{d}{dt}[t^k \omega_{i_1,\cdots,i_k}(tx)]dt \right) dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &= \sum_{i_1<\cdots<i_k} \omega_{i_1,\cdots,i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k} = \omega \end{aligned}$$

□

An example worthy of note which outrightly incorporates these discussed notions about differential forms and vector fields in physics is Maxwell's equations of electromagnetism. The setting is $\mathbb{R}^4$ (a space - time manifold), and performing relevant operations on the electromagnetic field as an exact differential 2-form yields mathematical interpretations of profound physical results. However, this is an illustration in Lorentzian geometry which differs from Riemannian geometry by way of the metric.

### 1.2.3 Integration over cubes and chains

Essentially, differential forms have to be integrated over domains where they are defined in $\mathbb{R}^n$. This gives rise to the use of singular k-cubes in domains of $\mathbb{R}^n$, which are suitable parametrizations of the domains for this purpose. A singular k-cube in $A \subseteq \mathbb{R}^n$ is a continuous function c mapping from $[0,1]^k$ to A. Any singular 1-cube is a curve, and singular 2-cubes are surfaces. Standard n-cubes in $\mathbb{R}^n$ are often denoted $I^n$ with $I^n : [0,1]^n \to \mathbb{R}^n$ defined by $I^n(x) = x$ for $x \in [0,1]^n$.
A linear combination $\sum_{i\in I\subseteq \mathbf{N}} a_i c_i$ ; $a_i \in \mathbf{Z}$ of singular k-cubes $c_i$ is referred to as a singular k-chain. Each singular k-chain $c$ has a boundary denoted $\partial c$ which is a (k-1) chain. To get the general formula of $\partial c$ for an n-chain $c$, we first formulate $\partial I^n$. For $i : 1 \leq i \leq n$, define the following singular (n-1)

cubes.

1) $I^n{}_{(i,0)}(x) = I^n(x_1, \cdots, x_{i-1}, 0, x_i, \cdots, x_{n-1}) = (x_1, \cdots, x_{i-1}, 0, x_i, \cdots, x_{n-1})$

2) $I^n{}_{(i,1)}(x) = I^n(x_1, \cdots, x_{i-1}, 1, x_i, \cdots, x_{n-1}) = (x_1, \cdots, x_{i-1}, 1, x_i, \cdots, x_{n-1})$ for $x \in [0,1]^{n-1}$.

$I^n{}_{(i,0)}$ is called the (i,0)-face of $I^n$ and $I^n{}_{(i,1)}$ the (i,1)-face.

Now, $\partial I^n := \sum_{i=1}^n \sum_{\alpha=0,1} (-1)^{i+\alpha} I^n{}_{(i,\alpha)}$.

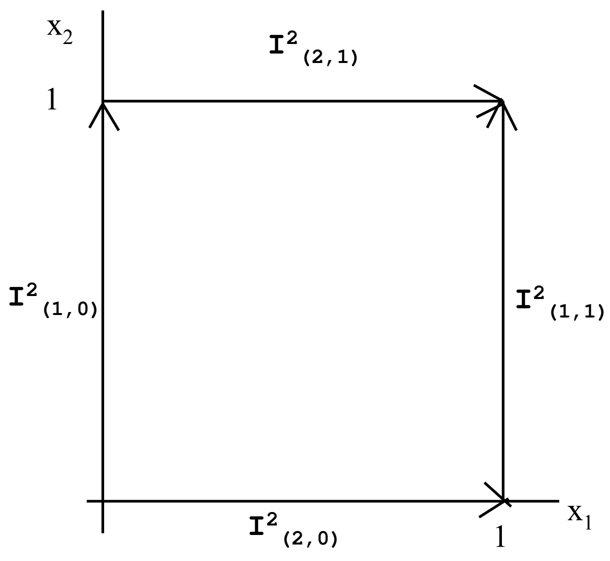
For a singular n-cube c; we define its $(i,\alpha)$-face, $c_{(i,\alpha)} = c \circ (I^n{}_{(i,\alpha)})$ so that

$$\partial c := \sum_{i=1}^{n} \sum_{\alpha=0,1} (-1)^{i+\alpha} c_{(i,\alpha)}.$$

Finally, the boundary of an n-chain $\sum\limits_{i \in I \subseteq \mathbf{N}} a_i c_i$ is given by

$$\partial \left( \sum_{i \in I} a_i c_i \right) = \sum_{i \in I} a_i \partial(c_i).$$

In $\mathbb{R}^2$ for instance, the boundary of $I^2$ is depicted as follows.




$\partial I^2$ can be described as the sum of four singular 1-cubes arranged counterclockwise around the boundary of $[0,1]^2$.

A property of the boundary operator $\partial$ is $\partial^2 = 0$ which is to say $\partial(\partial c) = 0$ for any singular n-chain $c$. Other properties and relations derived from the boundary operator are highlighted next in classical theorems by Stokes and Green. The orientations of domains of integration will also be considered, without which integrands obtained over singular k-cubes can only be guaranteed to be accurate up to sign.

## 1.3 Classical theorems of Green and Stokes

If $\omega$ is a k-form on $[0,1]^k$, then $\omega = f dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ for a unique 0-form $f$. We then have

$$\int_{[0,1]^k} \omega = \int_{[0,1]^k} f dx_1 \wedge \cdots \wedge dx_k = \int_{[0,1]^k} f(x_1, \cdots, x_k) dx_1 \cdots dx_k$$

For $\omega$ a k-form on $A \subseteq \mathbb{R}^n$ and c a singular k-cube in A, we define

$$\int_c \omega := \int_{[0,1]^k} c^* \omega$$

recalling that $c^*\omega$ is an induced k-form on $[0,1]^k$. The integral of a form $\omega$ over a k-chain $c = \sum_{i \in I} a_i c_i$ is given by $\int_c \omega = \sum_{i \in I} a_i \int_{c_i} \omega$.
The integral of a 1-form over a 1-chain is called a line integral and the integral of a 2-form over a singular 2-chain is called a surface integral.

Hitherto observations made permit a clear breakdown of the proof of a theorem by Stokes, which is popularly recognized as the fundamental theorem of calculus in higher dimensions.

**Theorem 1.3.1.** ***(Stokes' Theorem (a))***

*If $\omega$ is a (k-1) form on an open subset $A \subseteq \mathbb{R}^n$ and c is a k-chain in A, then $\int_c d\omega = \int_{\partial c} \omega$.*

*Proof.* We first take $c$ to be the standard k-cube $I^k$, and $\omega$ to be a (k-1) form on $[0,1]^k$.
In this case, $\omega$ can be written as the sum of (k-1) forms of the type

$$f dx_1 \wedge \cdots \wedge \underline{dx_i} \wedge \cdots \wedge dx_k$$

(the strikethrough beneath $dx_i$ indicates that this 1-form is excluded from the term), and we sufficiently prove the theorem for each of these. Note that

$$\int_{[0,1]^{k-1}} {I^k}_{(j,\alpha)}{}^* (f dx_1 \wedge \cdots \wedge \underline{dx_i} \wedge \cdots \wedge dx_k) = \delta_{ij} \int_{[0,1]^k} f(x_1, \cdots, \alpha, \cdots, x_k) dx_1 \cdots dx_k,$$

where $\delta_{ij} = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$

Thus,

$$\int_{\partial I^k} f dx_1 \wedge \cdots \wedge \underline{dx_i} \wedge \cdots \wedge dx_k$$
$$= \sum_{j=1}^{k} \sum_{\alpha=0,1} (-1)^{j+\alpha} \int_{[0,1]^{k-1}} {I^k}_{(j,\alpha)}{}^{*}(f dx_1 \wedge \cdots \wedge \underline{dx_i} \wedge \cdots \wedge dx_k)$$
$$= (-1)^{i+1} \int_{[0,1]^k} [f(x_1, \cdots, 1, \cdots, x_k) - f(x_1, \cdots, 0, \cdots, x_k)] dx_1 \cdots dx_k$$

Besides,

$$\int_{I^k} d(f dx_1 \wedge \cdots \wedge \underline{dx_i} \wedge \cdots \wedge dx_k)$$
$$= \int_{[0,1]^k} D_i f dx_i \wedge dx_1 \wedge \cdots \wedge \underline{dx_i} \wedge \cdots \wedge dx_k$$
$$= (-1)^{i-1} \int_{[0,1]^k} D_i f$$
$$= (-1)^{i-1} \int_0^1 \cdots \int_0^1 D_i f(x_1, \cdots, x_k) dx_i dx_1 \cdots \underline{dx_i} \cdots dx_k$$ (by Fubini's theorem)
$$= (-1)^{i-1} \int_0^1 \cdots \int_0^1 [f(x_1, \cdots, 1, \cdots, x_k) - f(x_1, \cdots, 0, \cdots, x_k)] dx_1 \cdots dx_i \cdots dx_k$$
(by the fundamental theorem of calculus in one-dimension)
$$= (-1)^{i+1} \int_{[0,1]^k} [f(x_1, \cdots, 1, \cdots, x_k) - f(x_1, \cdots, 0, \cdots, x_k)] dx_1 \cdots dx_k$$

Hence,

$$\int_{I^k} d\omega = \int_{\partial I^k} \omega.$$

Now, let c be an arbitrary singular k-cube, then

$$\int_c d\omega = \int_{I^k} c^*(d\omega) = \int_{I^k} d(c^*\omega) = \int_{\partial I^k} c^*\omega = \int_{\partial c} \omega.$$

Finally, if $c$ is a k-chain $\sum_{i \in I} a_i c_i$, then

$$\int_c d\omega = \sum_{i \in I} a_i \int_{c_i} d\omega = \sum_{i \in I} a_i \int_{\partial c_i} \omega = \int_{\partial c} \omega.$$

□

Before presenting the other theorems, we briefly view the structures of fields and forms on differentiable manifolds.

Let M be a k-dimensional manifold in $\mathbb{R}^n$ and the local chart around a point $p \in M$ be $(U, \phi)$. Then we can define a local coordinate system $\phi^{-1} : V \to \mathbb{R}^n$ ($V \subseteq \mathbb{R}^k$ is open) around $p = \phi^{-1}(a)$ for some $a \in V$.
The k-dimensional vector space $\phi_*^{-1}(\mathbb{R}^k_a)$ is denoted $T_pM$, and is called the tangent space of M at p. This space is independent of which local coordinate system is used to derive it. A function which assigns a vector in $T_pM$ to each point $p \in M$ is called a vector field on M. A function which assigns an alternating k-tensor in $\bigwedge^k(T_pM)$ to each $p \in M$ is called a k-form on M. Hence, given a vector field F on M, $F : M \to \bigcup_{p\in M} T_pM$ and a 1-form $\omega_p : T_pM \to \mathbb{R}$, we may obtain the composition $\omega_p(F) = \omega \circ F(p)$ which is a mapping from M to $\mathbb{R}$. Inadvertently, differential 1-forms constitute the dual to vector fields on a given manifold.

If $f : W \subseteq \mathbb{R}^k \to \mathbb{R}^n$ is a coordinate system, $\omega$ a k-form on M, then $f^*\omega$ is a k-form on W and we say $\omega$ is differentiable if $f^*\omega$ is. A k-form $\omega$ can be written $\omega = \sum_{i_1<\cdots<i_k} \omega_{i_1} \cdots \omega_{i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$.

Since the functions $\omega_{i_1}, \cdots, \omega_{i_k}$ may be defined only on M, the previous definition given for $d\omega$ may not be valid here, as $D_j(\omega_{i_1}, \cdots, \omega_{i_k})$ would have no meaning. However, the relation $f^*(d\omega) = d(f^*\omega)$ still holds, so we define the differential of $\omega$ as $d\omega = (f^{-1})^*(d(f^*\omega))$.

### 1.3.1 Orientable Manifolds

It is often important to choose, if possible, an orientation $\mu_p$ for each tangent space $T_pM$ of a manifold M. These choices are called consistent if, given a coordinate system $f : W \to \mathbb{R}^n$ and $a, b \in W$, then we have

$$[f_*((e_1)_a), \cdots, f_*((e_k)_a)] = \mu_{f(a)} \iff [f_*((e_1)_b), \cdots, f_*((e_k)_b)] = \mu_{f(b)}$$

If orientations $\mu_p$ have been chosen consistently and $f : W \to \mathbb{R}^n$ is a coordinate system such that $[f_*((e_1)_a), \cdots, f_*((e_k)_a)] = \mu_{f(a)}$ for one and hence every $a \in W$, then $f$ is called orientation - preserving. If $f$ is not orientation - preserving and $T : \mathbb{R}^k \to \mathbb{R}^k$ is a linear transformation such that $detT$ is negative, then $f \circ T$ is orientation - preserving. Hence, as long as orientations can be chosen consistently, there exists an orientation - preserving coordinate system around each point.

Suppose that $f$ and $g$ are orientation - preserving and $p = f(a) = g(b)$,

then
$[f_*((e_1)_a), \cdots, f_*((e_k)_a)] = \mu_p = [g_*((e_1)_b), \cdots, g_*((e_k)_b)]$. Therefore,
$[(g^{-1} \circ f)_*((e_1)_a), \cdots, (g^{-1} \circ f)_*((e_k)_a)] = [(e_1)_b, \cdots, (e_1)_b]$ so that
$det(g^{-1} \circ f)' > 0$.

A manifold for which orientations $\mu_p$ can be chosen consistently is orientable and a choice for $\mu_p$ is called an orientation of the manifold. A manifold M together with an orientation $\mu$ is called an oriented manifold.

One of the most known examples of a non-orientable manifold in $\mathbb{R}^3$ is the Möbius strip.

***Manifolds with Boundary***

If we have $M \subseteq \mathbb{R}^n$ to be a k-dimensional manifold - with - boundary, then for each point p $\in$ M, either

1. there exist open sets U and V with $p \in U \subseteq \mathbb{R}^n, V \subseteq \mathbb{R}^n$ and a diffeomorphism $\phi : U \to V$ such that $\phi(U \cap M) = V \cap (\mathbb{R}^k \times \{0\})$, *OR*
2. there exist open sets U and V, with $p \in U \subseteq \mathbb{R}^n, V \subseteq \mathbb{R}^n$ and a diffeomorphism $\phi : U \to V$ such that
$\phi(U \cap M) = V \cap (\mathbf{H}^k \times \{0\}) = \{y \in V : y_k \geq 0 \text{ and } y_{k+1} = \cdots = y_n = 0\}$,
and $\phi(p)$ has $k^{th}$ component equal to 0.

$\mathbf{H}^k = \{x \in \mathbb{R}^k : x_k \geq 0\}$ is called a half-space of $\mathbb{R}^k$.

Conditions (1) and (2) cannot be satisfied by the same point $p \in M$. Assuming on the contrary that there is a point which satisfies (1) and (2), then there would exist diffeomorphisms $\phi_1 : U_1 \to V_1$ and $\phi_2 : U_2 \to V_2$ such that $\phi_1(U_1 \cap M) = V_1 \cap \mathbb{R}^k$ and $\phi_2(U_2 \cap M) = V_2 \cap \mathbf{H}^k, \phi_2^k(p) = 0$.
The set $\phi_1(U_1 \cap U_2)$ would then be an open subset in $\mathbb{R}^k$ mapped onto $\phi_2(U_1 \cap U_2)$ by the diffeomorphism $\phi_2 \circ {\phi_1}^{-1}$. Since $\phi_2^k(p) = 0$, the set $\phi_2(U_1 \cap U_2)$ then contains a point from $\partial \mathbf{H}^k = \mathbb{R}^{k-1}$ and so it cannot be open in $\mathbb{R}^k$. This is a contradiction to the inverse function theorem.

A point $p \in M$ which satisfies (2) is called a boundary point of M and we denote by $\partial M$ the boundary of M, which is the set of all boundary points of M. If M is a k-dimensional manifold with boundary, then $\partial M$ is a (k - 1) dimensional submanifold without boundary. Let $M^* \supset M$ be a smooth k-dimensional manifold extended from M at its boundary. If $p \in \partial M$, then $T_p(\partial M)$ is a (k - 1) dimensional subspace of the k-dimensional space $T_p M^*$.

As a result, there are exactly 2 unit vectors in $T_pM^*$ which are perpendicular to $T_p(\partial M)$. If $(v_1, \cdots, v_k)$ is an orthonormal basis for $T_pM^*$ such that $(v_1, \cdots, v_{k-1})$ is a basis for $T_p(\partial M)$, then $v_k \in T_pM^*$ is one of the unit vectors perpendicular to $T_p(\partial M)$ and the other clearly is $-v_k$.

If $f : W \to \mathbb{R}^n$ is a coordinate system with $W \subseteq \mathbf{H}^k$ and $f(0) = p \in \partial M$, then only one of these unit vectors is $f_*(v_0)$ for some $v_0 \in W$ with $(v_0)_k < 0$. This unit vector is called the outward unit normal $n(p)$ and it is independent of the coordinate system $f$ used to obtain it.

Suppose that $\mu$ is an orientation of the k-dimensional manifold - with - boundary M. If $p \in \partial M$, we choose $v_1, \cdots, v_{k-1} \in T_p(\partial M)$ so that we have $[n(p), v_1, \cdots, v_{k-1}] = \mu_p$. If also $[n(p), w_1, \cdots, w_{k-1}] = \mu_p$ then both $[v_1, \cdots, v_{k-1}]$ and $[w_1, \cdots, w_{k-1}]$ are the same orientation for $T_p(\partial M)$, either of which is denoted by $(\partial\mu)_p$. If M is orientable, then $\partial M$ is also orientable and an orientation $\mu$ for M determines an orientation $\partial\mu$ for $\partial M$ called the induced orientation.

The ellipsoid $M_{(n)}$, as we recall from definition 1.1.3, is an n-dimensional manifold in $\mathbb{R}^{n+1}$ without boundary and it is the boundary for the (n+1) - dimensional manifold with boundary

$$L_{(n+1)} := \left\{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} : \frac{x_1^2}{a^2} + \frac{x_2^2}{a^2} + \cdots + \frac{x_n^2}{a^2} + \frac{x_{n+1}^2}{b^2} \leq 1\right\}$$

of $\mathbb{R}^{n+1}$. As such, if for $p \in M_{(n)}$ we have $[v_1, \cdots, v_n] = \mu_p$, we obtain the outward unit normal to $M_{(n)}$ at p; $\psi(p) \in \mathbb{R}^{n+1}{}_p$ so that $\psi(p)$ is a unit vector prependicular to $T_pM_{(n)}$ and $[\psi(p), v_1, \cdots, v_n]$ is the orientation of $\mathbb{R}^{n+1}{}_p$ which induces $\mu_p$. Note that for an interior point $a \in L_{(n+1)}$, the vector space $\mathbb{R}^{n+1}{}_a$ coincides with $T_aL_{(n+1)}$. A direct explanation for the orientability of $M_{(n)}$ is drawn from an alternative definition given as follows.

Let $f_1, \cdots, f_{n-k} : U \to \mathbb{R}$ be smooth functions defined on an open subset $U \subseteq \mathbb{R}^n$ with $df_1 \wedge \cdots \wedge df_{n-k} \neq 0$ at each point. Then the k-dimensional manifold $M_k := \{x \in U : f_1(x) = \cdots = f_{n-k}(x) = 0\}$ is orientable.

Let
$f : \mathbb{R}^{n+1} \backslash \{0\} \to \mathbb{R}$;
$x = (x_1, \cdots, x_{n+1}) \mapsto f(x) = \|bx\|^2 - a^2b^2 + (a^2 - b^2)x_{n+1}^2$
Then
$df = 2b^2x_1dx_1 + 2b^2x_2dx_2 + \cdots + 2b^2x_ndx_n + 2a^2x_{n+1}dx_{n+1} \neq 0$ on $\mathbb{R}^{n+1}\backslash\{0\}$.

Hence, $M_{(n)} = \{x \in \mathbb{R}^{n+1} \backslash \{0\} : f(x) = 0\}$ is orientable.

We now state further theorems utilizing the concepts of boundaries of manifolds and their orientations.

**Theorem 1.3.2.** ***(Stokes' Theorem (b))***

*If M is a compact oriented k-dimensional manifold - with - boundary and $\omega$ is a (k - 1) form on M, then $\int_M d\omega = \int_{\partial M} \omega$ where $\partial M$ is given the induced orientation.*

The proof of this theorem incorporates a standard tool required in the theory of integration called partitions of unity.

**Lemma 1.3.3.**
*For $A \subseteq \mathbb{R}^n$ and O an open cover of A, there is a collection $\Phi$ of $C^\infty$ functions $\varphi$ defined in an open set containing A called a $C^\infty$ partition of unity for A subordinate to the cover O, with the following properties:*

*(1) For each $x \in A$ we have $0 \leq \varphi(x) \leq 1$.*

*(2) For each $x \in A$ there is an open set V containing x such that all but finitely many $\varphi \in \Phi$ are 0 on V.*

*(3) For each $x \in A$, we have $\sum_{\varphi \in \Phi} \varphi(x) = 1$. By (2) for each x this sum is finite in some open set containing x.*

*(4) For each $\varphi \in \Phi$ there is an open set U in O such that $\varphi = 0$ outside of some closed set contained in U.*

*Proof of Theorem 1.3.2* Commencing the proof this theorem, suppose that there is an orientation - preserving singular k-cube c in $M - \partial M$ such that $\omega = 0$ outside of $c([0,1]^k)$. By Theorem 1.3.1 and the definition of $d\omega$ we have

$$\int_c d\omega = \int_{[0,1]^k} c^*(d\omega) = \int_{[0,1]^k} d(c^*\omega) = \int_{\partial I^k} c^*\omega = \int_{\partial c} \omega.$$

Then,

$$\int_M d\omega = \int_c d\omega = \int_{\partial c} \omega = 0,$$

since $\omega = 0$ on $\partial c$. On the other hand, $\int_{\partial M} \omega = 0$ since $\omega = 0$ on $\partial M$.

Suppose next that there is an orientation-preserving singular k-cube in M such that $c_{(k,0)}$ is the only face in $\partial M$, and $\omega = 0$ outside of $c([0,1]^k)$. Then

$$\int_M d\omega = \int_c d\omega = \int_{\partial c} \omega = \int_{\partial M} \omega.$$

We may now consider the general case. There is an open cover $O$ of M and a partition of unity $\Phi$ for M subordinate to $O$ such that for each $\varphi \in \Phi$ the form $\varphi \cdot \omega$ is of one of the two sorts just considered. We have

$$0 = d(1) = d\left(\sum_{\varphi \in \Phi} \varphi\right) = \sum_{\varphi \in \Phi} d\varphi,$$

so that

$$\sum\nolimits_{\varphi \in \Phi} d\varphi \wedge \omega = 0.$$

Since M is compact, this is a finite sum and we have

$$\sum_{\varphi \in \Phi} \int_M d\varphi \wedge \omega = 0.$$

Therefore,

$$\begin{aligned}
\int_M d\omega &= \sum_{\varphi \in \Phi} \int_M \varphi \cdot d\omega \\
&= \sum_{\varphi \in \Phi} \int_M d\varphi \wedge \omega + \varphi \cdot d\omega \\
&= \sum_{\varphi \in \Phi} \int_M d(\varphi \cdot \omega) \\
&= \sum_{\varphi \in \Phi} \int_{\partial M} \varphi \cdot \omega \\
&= \int_{\partial M} \omega.
\end{aligned}$$

Now, we give some practical versions of Stokes' Theorem.

**Theorem 1.3.4.** ***Green's Theorem***

*Let $M \subset \mathbb{R}^2$ be a compact 2-dimensional manifold - with - boundary. Suppose that $\alpha, \beta : M \to \mathbb{R}$ are differentiable. Then*

$$\int_{\partial M} \alpha dx + \beta dy = \int_M (D_1\beta - D_2\alpha) dx \wedge dy = \int\int_M (\frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y}) dxdy,$$

*where $M$ is given the usual orientation and $\partial M$ the induced orientation, otherwise called the counterclockwise orientation.*

*Proof.* We find the differential of the 1-form $(\alpha dx + \beta dy)$ to be

$$\begin{aligned} d(\alpha dx + \beta dy) &= d\alpha \wedge dx + d\beta \wedge dy \\ &= (D_1\alpha dx + D_2\alpha dy) \wedge dx + (D_1\beta dx + D_2\beta dy) \wedge dy \\ &= D_2\alpha dy \wedge dx + D_1\beta dx \wedge dy \\ &= (D_1\beta - D_2\alpha) dx \wedge dy \end{aligned}$$

We now apply theorem 1.3.2 directly to get

$$\begin{aligned} \int_{\partial M} \alpha dx + \beta dy &= \int_M d(\alpha dx + \beta dy) \\ &= \int_M (D_1\beta - D_2\alpha) dx \wedge dy \\ &= \int\int_M (\frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y}) dxdy \end{aligned}$$

□

**Theorem 1.3.5.** ***(Stokes' Theorem (c))***

*Let $M \subset \mathbb{R}^3$ be a compact oriented two-dimensional manifold - with - boundary and $n$ the unit outward normal on $M$ determined by the orientation of $M$. Let $\partial M$ have the induced orientation. Let $G$ be the vector field on $\partial M$ with $ds(G) = 1$ and $F$ be a differentiable vector field in an open set containing*

*M. Then* $\int_M \langle(\nabla \times F), n\rangle dA = \int_{\partial M}\langle F, G\rangle ds$
*(dA and ds are respectively referred to as element of area and element of arclength.)*

*Proof.* Define $\eta$ on M by $\eta = F^1dx + F^2dy + F^3dz$. Recall the curl of F, $\nabla\times F$ respectively has components $D_2F^3 - D_3F^2, D_3F^1 - D_1F^3, D_1F^2 - D_2F^1$. For a two-dimensional manifold, the element of volume is the element of area $dA \in \bigwedge^2(T_pM)$ and

$$dA(v,w) = det\begin{pmatrix} v \\ w \\ n(p)\end{pmatrix} \forall v, w \in T_pM,$$

where $n(p)$ is the outward unit normal, since $dA(v,w)$ is 1 if $v$ and $w$ form an orthonormal basis for $T_pM$ with $[v,w] = \mu_p$.
Note that $dA(v,w) = \langle v \times w, n(p)\rangle$.
Let $\xi \in \mathbb{R}^3_p$, observing that $v \times w = \alpha.n(p), \quad \alpha = \pm\|v \times w\| \in \mathbb{R}$.
$\langle\xi, n(p)\rangle dA(v,w) = \langle\xi, n(p)\rangle\alpha = \langle\xi, \alpha n(p)\rangle = \langle\xi, v \times w\rangle$

The above scalar triple product equals

$$\begin{vmatrix} \xi_1 & \xi_2 & \xi_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \xi_1(v_2w_3 - v_3w_2) - \xi_2(v_1w_3 - v_3w_1) + \xi_3(v_1w_2 - v_2w_1).$$

$$dy \wedge dz(v,w) = 2Alt(dy \otimes dz(v,w)) = 2.\frac{1}{2!}(v_2w_3 - v_3w_2)$$
$$\implies \xi_1 dy \wedge dz(v,w) = \xi_1(v_2w_3 - v_3w_2)$$

Likewise,
$\xi_2dz \wedge dx(v,w) = \xi_2(v_3w_1 - v_1w_3)$ and $\xi_3dx \wedge dy(v,w) = \xi_3(v_1w_2 - v_2w_1)$.

Thus,

$$\langle\xi, n(p)\rangle dA = \xi_1dy \wedge dz + \xi_2dz \wedge dx + \xi_3dx \wedge dy$$

and
$\langle(\nabla \times F), n\rangle dA$
$= (D_2F^3 - D_3F^2)dy \wedge dz + (D_3F^1 - D_1F^3)dz \wedge dx + (D_1F^2 - D_2F^1)dx \wedge dy$
$= d\eta.$

Also, since ds(G) = 1 on $\partial M$ we have $G^1ds = dx, G^2ds = dy, G^3ds = dz$. These equations are easily checked by applying both sides to G(p) for $p \in \partial M$, since G(p) is a basis for $T_p(\partial M)$. Therefore, on $\partial M$ we have

$$\begin{aligned}
\langle F, G\rangle ds &= F^1G^1ds + F^2G^2ds + F^3G^3ds \\
&= F^1dx + F^2dy + F^3dz \\
&= \eta
\end{aligned}$$

By theorem 1.3.2, we get

$$\begin{aligned}
\int_M \langle(\nabla \times F), n\rangle dA &= \int_M d\eta \\
&= \int_{\partial M} \eta \\
&= \int_{\partial M} \langle F, G\rangle ds
\end{aligned}$$

□

### 1.3.2 Riemannian Manifolds

As illustrations for the theoretical content of this chapter, we will consider specific examples of differential forms on Riemannian manifolds in chapter 2. We define a Riemannian manifold as a manifold M, equipped with a Riemannian metric $g$. At each point p of M, the metric $g_p$ must have the following properties.

1. $g_p : (T_pM \times T_pM) \to \mathbb{R}$ is bilinear.
2. $g_p(v, w) = g_p(w, v) \ \forall v, w \in T_pM$, which is to say $g_p$ is symmetric.
3. $g_p(v, v) > 0 \ \forall v \in T_pM : v \neq 0$, which is to say $g_p$ is positive definite.
4. The coefficients $g_{ij}$ in every local chart

$$g_p = \sum_{i,j} g_{ij}(p) \cdot dx_i|_p \otimes dx_j|_p$$

are differentiable functions, where $dx_i \otimes dx_j(a, b) = dx_i(a) \cdot dx_j(b)$.

For further remarks on the Riemannian metric tensor, see section 3 of chapter 2 on manifolds in higher dimensions.

CHAPTER 2

# EXAMPLES OF DIFFERENTIAL FORMS ON RIEMANNIAN MANIFOLDS

## 2.1 Winding form and volume element associated to ellipsoids in $\mathbb{R}^2$ and in $\mathbb{R}^3$

### 2.1.1 Differential forms on the 1-dimensional ellipsoid

Consider the ellipse $M_{(1)} \subset \mathbb{R}^2$ given by $\left\{(x, y) \in \mathbb{R}^2 : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1\right\}$ where $a$ and $b$ are positive real constants. The eccentricity $\varepsilon$ of ellipse $M_{(1)}$ is given by $\sqrt{1 - (\frac{a}{b})^2}$ for $b > a$ and $\varepsilon = \sqrt{1 - (\frac{b}{a})^2}$ for $a > b$. As a one-dimensional manifold, we can only define one-forms (and zero forms) on it. The first form which we will apply here is referred to as the winding form, which we denote $\omega_1$. It is given by

$$\omega_1 \;=\; \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy$$

and defined on $\mathbb{R}^2 \backslash \{0\} \supset M_{(1)}$.

This one-form is an example of a closed form which is not exact. Indeed, the differential of $\omega_1$ is

$$\begin{aligned}
d\omega_1 &= \frac{(x^2+y^2)D_1(-y)+yD_1(x^2+y^2)}{(x^2+y^2)^2}dx\wedge dx \\
&+ \frac{(x^2+y^2)D_2(-y)+yD_2(x^2+y^2)}{(x^2+y^2)^2}dy\wedge dx \\
&+ \frac{(x^2+y^2)D_1(x)-xD_1(x^2+y^2)}{(x^2+y^2)^2}dx\wedge dy \\
&+ \frac{(x^2+y^2)D_2(x)-xD_2(x^2+y^2)}{(x^2+y^2)^2}dy\wedge dy \\
&= \frac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2}dy\wedge dx + \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2}dx\wedge dy \\
&= \frac{x^2-y^2}{(x^2+y^2)^2}dx\wedge dy + \frac{y^2-x^2}{(x^2+y^2)^2}dx\wedge dy \\
&= 0
\end{aligned}$$

Thus, $\omega_1$ is closed.

$M_{(1)}$ can be parametrized by the function $f$;

$$\begin{aligned}
f:\ [0,2\pi] &\longrightarrow \mathbb{R}^2 \\
\alpha &\longmapsto f(\alpha)=(a\cos\alpha, b\sin\alpha),
\end{aligned}$$

or by the singular 1-cube $\gamma$ in $M_{(1)}$;

$$\begin{aligned}
\gamma:\ [0,1] &\longrightarrow \mathbb{R}^2 \\
t &\longmapsto \gamma(t)=(a\cos(2\pi t), b\sin(2\pi t))
\end{aligned}$$

The parametrization $f$ is orientation - preserving when $\mathbb{R}^2$ is endowed with its usual orientation because as $\alpha$ runs from 0 to $2\pi$, the points $f(\alpha)\in M_{(1)}$ run counterclockwise about the origin starting from $(a,0)$. Likewise, $\gamma$ is orientation - preserving because the map $T:\mathbb{R}\to\mathbb{R}$ with $T(t)=2\pi t$ is orientation - preserving and $\gamma=f\circ T$. Hence,

$$
\begin{aligned}
\int_\gamma \omega_1 &= \int_{[0,1]} \gamma^*\omega_1 \\
&= \int_0^1 \omega_1(\gamma) \\
&= \int_0^1 \left( \frac{2\pi ab \sin^2(2\pi t) + 2\pi ab \cos^2(2\pi t)}{a^2\cos^2(2\pi t) + b^2 \sin^2(2\pi t)} \right) dt \\
&= 2\pi \int_0^1 \left( \frac{ab}{a^2\cos^2(2\pi t) + b^2 \sin^2(2\pi t)} \right) dt \\
&= \int_0^{2\pi} \left( \frac{ab}{a^2\cos^2\alpha + b^2\sin^2\alpha} \right) d\alpha \\
&= ab \int_0^{2\pi} \left( \frac{1}{b^2 + (a^2 - b^2)\cos^2\alpha} \right) d\alpha
\end{aligned}
$$

We may apply the Cauchy residues theorem to evaluate this integral, which would involve complexification of the $\mathbb{R}^2$ space. Briefly, to evaluate $\int_0^{2\pi} R(\sin\alpha, \cos\alpha) d\alpha$, we compute $2\pi \sum_{|z_i|<1} Rez(g; z_i)$ for all singular points $z_i$, where $g(z) = \frac{1}{z} R(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2})$.

In our case,

$$
\begin{aligned}
g(z) &= \frac{1}{z} \frac{1}{b^2 + (a^2 - b^2)((z + z^{-1})/2)^2} \\
&= \frac{4z}{(a^2 - b^2)z^4 + (2a^2 + 2b^2)z^2 + a^2 - b^2}
\end{aligned}
$$

The singular points of $g$ are the four roots of its denominator which are

$z_1 := i\sqrt{\frac{|a-b|}{a+b}}, z_2 := -i\sqrt{\frac{|a-b|}{a+b}}, z_3 := i\sqrt{\frac{a+b}{|a-b|}}, z_4 := -i\sqrt{\frac{a+b}{|a-b|}}$.

They are all simple poles and

$|z_1| = |z_2| = \sqrt{\frac{|a-b|}{a+b}}$

$|z_3| = |z_4| = \sqrt{\frac{a+b}{|a-b|}}$

Since $a$ and $b$ are positive, $|z_1| = |z_2| < 1 < |z_3| = |z_4|$.

$$\begin{aligned}
\int_0^{2\pi} \left( \frac{ab}{a^2\cos^2\alpha + b^2\sin^2\alpha} \right) d\alpha &= 2\pi ab(Rez(g; z_1) + Rez(g; z_2)) \\
&= 2\pi ab(\underset{z\to z_1}{lim}(z - z_1).g(z) + \underset{z\to z_2}{lim}(z - z_2).g(z)) \\
&= 2\pi ab(A + B)
\end{aligned}$$

where, $A = \lim_{z\to z_1} \frac{4z}{(a^2 - b^2)(z - z_2)(z - z_3)(z - z_4)} = \frac{1}{2ab}$ and

$$B = \lim_{z\to z_2} \frac{4z}{(a^2 - b^2)(z - z_1)(z - z_3)(z - z_4)} = \frac{1}{2ab}.$$

Hence, the integral of the winding form $\omega_1$ over the one dimensional manifold or curve $M_{(2)}$, equals

$$\int_\gamma \omega_1 = 2\pi.$$

The integral of the winding form along a closed curve in $\mathbb{R}^2$ surrounding the origin is a measure of how often the curve turns around the origin. Many sources also give the winding form the notation $d\theta$, where $\theta$ is an angle measure.

Assume that $\omega_1$ were the differential of a smooth function, say h. Then

$$\begin{aligned}
\int_\gamma \omega_1 &= \int_{[0,1]} \gamma^*\omega_1 = \int_0^1 \gamma^*(dh) \\
&= \int_0^1 d(h\circ\gamma) \\
&= \int_0^1 \frac{d(h\circ\gamma)(t)}{dt} dt \\
&= h(\gamma(1)) - h(\gamma(0))
\end{aligned}$$

Hence, the line integral of an exact 1-form over a closed curve vanishes since $\gamma(1)$ equals $\gamma(0)$ in this case. This is how we see that the winding form $\omega_1$ is not exact, as its integral over the closed curve $M_{(1)}$ is not zero.

***Volume form on the one dimensional ellipsoid***

Another 1-form we can apply to $M_{(1)}$ is the volume form which in this case coincides with the element of arclength since $M_{(1)}$ is a closed curve in $\mathbb{R}^2$. Integrating the volume form over a manifold yields its Lebesgue measure. We denote the element of arclength $ds$; $(ds)^2 = (dx)^2 + (dy)^2$.

$$\begin{aligned}
\int_\gamma ds &= \int_{[0,1]} \gamma^* ds, \quad \text{(for the singular 1-cube } \gamma \text{ in } M_1 \text{ given in this section)} \\
&= \int_0^1 \sqrt{4\pi^2 a^2 \sin^2(2\pi t) + 4\pi^2 b^2 \cos^2(2\pi t)}\, dt \\
&= \int_0^{2\pi} \sqrt{a^2 \sin^2\alpha + b^2 \cos^2\alpha}\, d\alpha \\
&= b\int_0^{2\pi} \sqrt{1 + \left(\frac{a^2}{b^2} - 1\right) \sin^2\alpha}\, d\alpha \\
&= b\int_0^{2\pi} \sqrt{1 - \varepsilon^2 \sin^2\alpha}\, d\alpha \\
&= 4b\int_0^{\frac{\pi}{2}} \sqrt{1 - \varepsilon^2 \sin^2\alpha}\, d\alpha
\end{aligned}$$

The antiderivative of the above integrand cannot be expressed in terms of elementary functions. However, we recognize the final expression as a complete elliptic integral which precisely has the following power series expansion

$$2\pi b \sum_{n=0}^{\infty} \frac{-1}{2n-1} \left( \frac{(2n)!}{(2^n n!)^2} \right)^2 \varepsilon^{2n};$$

where $\varepsilon^2 = 1 - (\frac{a}{b})^2$.

By the ratio test, this series converges for all values of $\varepsilon \in \mathbb{R}$ for which $\varepsilon < 1$. The element of arclength is not exact (despite its given notation ds) since its integral over the closed curve $M_{(1)}$ is not zero.

### 2.1.2 Differential forms on the 2-dimensional ellipsoid

Now, for $a, b \in \mathbb{R}^+\backslash\{0\}$, consider the ellipsoid $M_{(2)} \subseteq \mathbb{R}^3$ given by

$$M_{(2)} = \left\{(x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1\right\}.$$

We may define differential forms up to the second order on $M_{(2)}$ because it is a two-dimensional manifold. Let us begin with a 2-form ($\omega_2$) which is analogous to the winding form $\omega_1$. The form $\omega_2$ is given as

$$\omega_2 = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{r^3}$$

where $r = \sqrt{(x^2 + y^2 + z^2)}$.

This 2-form is defined on $\mathbb{R}^3\backslash\{0\} \supset M_{(2)}$ and it is a closed form which is not exact. The differential of $\omega_2$ is

$$\begin{aligned}
d(\omega_2) &= \frac{r^3(3dx \wedge dy \wedge dz) - (xdy \wedge dz + ydz \wedge dx + zdx \wedge dy) \wedge d(r^3)}{r^6} \\
&= \frac{r^3(3dx \wedge dy \wedge dz) - 3r^2(xdy \wedge dz + ydz \wedge dx + zdx \wedge dy) \wedge d(r)}{r^6} \\
&= \frac{r^3(3dx \wedge dy \wedge dz) - 3r(x^2dx \wedge dy \wedge dz + y^2dy \wedge dz \wedge dx + z^2dz \wedge dx \wedge dy)}{r^6} \\
&= \frac{3r^3dx \wedge dy \wedge dz - 3r \cdot r^2dx \wedge dy \wedge dz}{r^6} = 0
\end{aligned}$$

As such, $\omega_2$ is a closed 2-form.

$M_{(2)}$ can be parametrized by the function $\Phi$;

$$\begin{aligned}
\Phi : \quad [0, 2\pi] \times [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] &\longrightarrow M_{(2)} \subset \mathbb{R}^3 \\
(\alpha, \beta) &\longmapsto (a \cos\alpha \cos\beta, a \sin\alpha \cos\beta, b \sin\beta),
\end{aligned}$$

or by the singular 2-cube c in $M_{(2)}$;

$$\begin{aligned}
c : \quad [0, 1] \times [0, 1] &\longrightarrow \mathbb{R}^3 \\
(t, u) &\longmapsto (a \cos(2\pi t) \sin(\pi u), a \sin(2\pi t) \sin(\pi u), -b \cos(\pi u)).
\end{aligned}$$

These parametrizations yield the following partial derivatives

$\dfrac{\partial\Phi}{\partial\alpha} := \Phi_*(e_1)_{(\alpha,\beta)} = (-a\sin\alpha\cos\beta, a\cos\alpha\cos\beta, 0);$

$\dfrac{\partial\Phi}{\partial\beta} := \Phi_*(e_2)_{(\alpha,\beta)} = (-a\cos\alpha\sin\beta, -a\sin\alpha\sin\beta, b\cos\beta);$

$\dfrac{\partial c}{\partial t} := c_*(e_1)_{(t,u)} = (-2\pi a\sin(2\pi t)\sin(\pi u), 2\pi a\cos(2\pi t)\sin(\pi u), 0);$

$\dfrac{\partial c}{\partial u} := c_*(e_2)_{(t,u)} = (\pi a\cos(2\pi t)\cos(\pi u), \pi a\sin(2\pi t)\cos(\pi u), b\pi\sin(\pi u)).$

Take the point $p_0 = (-a, 0, 0) \in M_{(2)}$ at which the outward unit normal $n(p_0)$ to the manifold is clearly $(-1, 0, 0)$ when $\mathbb{R}^3$ is endowed with its usual orientation. We have $\Phi(\pi, 0) = (-a, 0, 0) =: p_0$, with this point having no other pre-images under $\Phi$. Let $\nu_0 = (\pi, 0)$ so that $\Phi(\nu_0) = p_0$.

$\Phi_*(e_1)_{(\nu_0)} = (0, -a, 0)$ and $\Phi_*(e_2)_{(\nu_0)} = (0, 0, b)$.

If $\Phi$ is an orientation - preserving parametrization, then $[n(p_0), \Phi_*(e_1)_{(\nu_0)}, \Phi_*(e_2)_{(\nu_0)}]$ must be the usual orientation of $\mathbb{R}^3$.

$$Det\begin{pmatrix} n(p_0) \\ \Phi_*(e_1)_{(\nu_0)} \\ \Phi_*(e_2)_{(\nu_0)} \end{pmatrix} = \begin{vmatrix} -1 & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & b \end{vmatrix} = ab > 0$$

Hence, $\Phi$ is an orientation-preserving parametrization when $\mathbb{R}^3$ is endowed with its usual orientation and so is the succeeding singular two-cube c because in the affine transformation, $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$\begin{pmatrix} t \\ u \end{pmatrix} \longmapsto \begin{pmatrix} 2\pi & 0 \\ 0 & \pi \end{pmatrix}\begin{pmatrix} t \\ u \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{-\pi}{2} \end{pmatrix}$$

we see that $c = \Phi \circ T$ and the matrix $\begin{pmatrix} 2\pi & 0 \\ 0 & \pi \end{pmatrix}$ is positive definite. We may proceed to integrate $\omega_2$ over an orientation - preserving parametrization.

$$\int_c \omega_2 = \int_{[0,1]^2} c^*\omega_2 = \int_U \Phi^*\omega_2; \text{ where } U = [0, 2\pi] \times [\tfrac{-\pi}{2}, \tfrac{\pi}{2}].$$

With respect to the parametrization $\Phi$ we get the differential 1-forms:

$dx = -a\sin\alpha\cos\beta d\alpha - a\cos\alpha\sin\beta d\beta$ ;
$dy = a\cos\alpha\cos\beta d\alpha - a\sin\alpha\sin\beta d\beta$ ;
$dz = b\cos\beta d\beta$.

Hence by substituting in $\Phi^*\omega_2(e_1, e_2) = \omega_2(\Phi)(\Phi_*(e_1), \Phi_*(e_2))$, we have as our integrand

$$\int_U \Phi^*\omega_2 = \int_U \left( \frac{\eta_1 + \eta_2 + \eta_3}{(a^2\cos^2\beta + b^2\sin^2\beta)^{\frac{3}{2}}} \right)$$

where

$$\eta_1 = xdy \wedge dz = a\cos\alpha\cos\beta(ab\cos\alpha\cos^2\beta d\alpha \wedge d\beta)$$

$$\eta_2 = ydz \wedge dx = a\sin\alpha\cos\beta(-ab\sin\alpha\cos^2\beta d\beta \wedge d\alpha)$$

$$\eta_3 = zdx\wedge dy = b\sin\beta(a^2\sin^2\alpha\sin\beta\cos\beta d\alpha\wedge d\beta - a^2\cos^2\alpha\sin\beta\cos\beta d\beta\wedge d\alpha).$$

Factorizing trigonometric expressions, we then have

$$\int_U \left( \frac{a^2b\cos\beta(\cos^2\alpha\cos^2\beta + \sin^2\alpha\cos^2\beta + \sin^2\alpha\sin^2\beta + \cos^2\alpha\sin^2\beta)d\alpha \wedge d\beta}{(a^2\cos^2\beta + b^2\sin^2\beta)^{\frac{3}{2}}} \right)$$

$$= \int_U \left( \frac{a^2b\cos\beta(\cos^2\beta + \sin^2\beta)d\alpha \wedge d\beta}{(a^2\cos^2\beta + b^2\sin^2\beta)^{\frac{3}{2}}} \right)$$

$$= -\int_U \frac{a^2b\cos\beta}{(a^2\cos^2\beta + b^2\sin^2\beta)^{\frac{3}{2}}} d\beta \wedge d\alpha$$

$$= -\int_U d\left( \frac{b\sin\beta}{(a^2\cos^2\beta + b^2\sin^2\beta)^{\frac{1}{2}}} \right) \wedge d\alpha$$

$$= -\int_{\partial U} \frac{b\sin\beta}{(a^2\cos^2\beta + b^2\sin^2\beta)^{\frac{1}{2}}} d\alpha \qquad \text{by Theorem 1.3.2}$$

Finally, by setting

$$g(\alpha, \beta) = \frac{-\alpha b\sin\beta}{\sqrt{a^2\cos^2\beta + b^2\sin^2\beta}}$$

we get the above integrand to equal

$$g(\alpha, \tfrac{-\pi}{2})\big|_{\alpha=0}^{\alpha=2\pi} + g(\alpha, \tfrac{\pi}{2})\big|_{\alpha=0}^{\alpha=2\pi} + g(0, \beta)\big|_{\beta=\frac{-\pi}{2}}^{\beta=\frac{\pi}{2}} + g(2\pi, \beta)\big|_{\beta=\frac{-\pi}{2}}^{\beta=\frac{\pi}{2}}$$

$$= \alpha|_0^{2\pi} - \alpha|_0^{2\pi} + 0 - \left[ \frac{2\pi b\sin\beta}{\sqrt{a^2\cos^2\beta + b^2\sin^2\beta}} \right]_{\beta=\frac{-\pi}{2}}^{\beta=\frac{\pi}{2}} = -4\pi.$$

Observe that the closed 2-form $\omega_2$ is non-degenerate so that it is a symplectic form on $M_{(2)}$ and we may say that the pair $(M_{(2)}, \omega_2)$ is a symplectic manifold. The closed property is the final requirement of symplectic forms not mentioned in section 2 of chapter 1. However, $\omega_2$ should not be confused with the *canonical symplectic structure*, which is related to volume forms in symplectic geometry, differing from Riemannian geometry by way of the metric.

***Volume form on orientable surfaces in*** $\mathbb{R}^3$

The volume form on an orientable surface in $\mathbb{R}^3$ is usually also referred to as the element of area. We denote the element of area dA and describe how to obtain it for an oriented two-dimensional manifold or surface in $\mathbb{R}^3$.

Let M be our oriented surface in $\mathbb{R}^3$, $p \in M$ and the outward unit normal to M at $p$ be $n(p)$. As we have seen in the proof of theorem 1.3.5, defining $\omega \in \bigwedge^2(T_pM)$ by

$\omega(v_1, v_2) = det \begin{pmatrix} v_1 \\ v_2 \\ n(p) \end{pmatrix}$,

we get $\omega(v_1, v_2) = 1$ if $v_1$ and $v_2$ form an orthonormal basis for $T_pM$ with $[v_1, v_2] = \mu_p$. Hence $\omega$ is the element of area dA, and letting $v_1, v_2 \in T_pM$ arbitrarily, we have

$$dA(v_1, v_2) = \omega(v_1, v_2) = \langle (v_1 \times v_2), n(p) \rangle = \pm \| v_1 \times v_2 \|$$

since $n(p)$ is perpendicular to the vectors $v_1$ and $v_2$.

To compute the area of M, we integrate the 2-form dA over the surface which is equivalent to evaluating $\int_{[0,1]^2} c^*(dA)$ for an orientation-preserving singular 2-cube c in M. Let

$c : [0,1]^2 \longrightarrow M \subset \mathbb{R}^3$

$\nu \longmapsto c(\nu) = p = (c^1(\nu), c^2(\nu), c^3(\nu))$

Then,

$c^*(dA)((e_1)_\nu, (e_2)_\nu) = dA(c_*((e_1)_\nu), c_*((e_2)_\nu))$

$= \| c_*((e_1)_\nu) \times c_*((e_2)_\nu) \|$

$= \| (D_1c^1(\nu), D_1c^2(\nu), D_1c^3(\nu)) \times (D_2c^1(\nu), D_2c^2(\nu), D_2c^3(\nu)) \|$

$$= \begin{pmatrix} (D_2c^3(\nu)D_1c^2(\nu) - D_2c^2(\nu)D_1c^3(\nu))^2 + (D_2c^3(\nu)D_1c^1(\nu) - D_2c^1(\nu)D_1c^3(\nu))^2 \\ +(D_2c^2(\nu)D_1c^1(\nu) - D_2c^1(\nu)D_1c^2(\nu))^2 \end{pmatrix}^{\frac{1}{2}}$$

$$= \begin{pmatrix} ([D_1c^1(\nu)]^2 + [D_1c^2(\nu)]^2 + [D_1c^3(\nu)]^2)([D_2c^1(\nu)]^2 + [D_2c^2(\nu)]^2 + [D_2c^3(\nu)]^2) \\ -(D_1c^1(\nu)D_2c^1(\nu) + D_1c^2(\nu)D_2c^2(\nu) + D_1c^3(\nu)D_2c^3(\nu))^2 \end{pmatrix}^{\frac{1}{2}}$$

We may set $\nu = (t, u) \in [0,1] \times [0,1]$ in order to rewrite the above expression as
$\sqrt{\langle \frac{\partial c}{\partial t}, \frac{\partial c}{\partial t} \rangle \cdot \langle \frac{\partial c}{\partial u}, \frac{\partial c}{\partial u} \rangle - \langle \frac{\partial c}{\partial t}, \frac{\partial c}{\partial u} \rangle^2}$

The three inner products involved above constitute the so-called Riemannian structure of M. The Riemannian structure of an n-dimensional manifold P parametrized by $f : U \subseteq \mathbb{R}^n \to P; u = (u_1, u_2, ..., u_n) \mapsto f(u)$ is the set of all inner products $\langle \frac{\partial f}{\partial u_i}, \frac{\partial f}{\partial u_j} \rangle$ for $1 \leq i, j \leq n$. Henceforth, we denote $\langle \frac{\partial c}{\partial t}, \frac{\partial c}{\partial t} \rangle = \left\| \frac{\partial c}{\partial t} \right\|^2$ by E; $\langle \frac{\partial c}{\partial u}, \frac{\partial c}{\partial u} \rangle = \left\| \frac{\partial c}{\partial u} \right\|^2$ by G and $\langle \frac{\partial c}{\partial t}, \frac{\partial c}{\partial u} \rangle$ by F. Finally, the area of the surface M is obtained as

$$\int_0^1 \int_0^1 \sqrt{EG - F^2} dt du.$$

The Riemannian structure of $M_{(2)}$ is given as:
$E = 4\pi^2 a^2 \sin^2(\pi u)$
$F = 0$
$G = \pi^2 a^2 \cos^2(\pi u) + \pi^2 b^2 \sin^2(\pi u)$,
by imputing from the orientation-preserving 2-cube c in $M_{(2)}$ given earlier in this section.

Area $(M_{(2)})$
$= \int_0^1 \int_0^1 |2\pi a \sin(\pi u)| \sqrt{\pi^2 a^2 \cos^2(\pi u) + \pi^2 b^2 \sin^2(\pi u)} dt du$
$= 2\pi^2 a \int_0^1 |\sin(\pi u)| \sqrt{a^2 \cos^2(\pi u) + b^2 \sin^2(\pi u)} du$
$= 2\pi^2 a \int_0^1 \sin(\pi u) \sqrt{a^2 \cos^2(\pi u) + b^2 \sin^2(\pi u)} du$
$= 2\pi a \int_{-1}^1 \sqrt{a^2 \vartheta^2 + b^2(1 - \vartheta^2)} d\vartheta \qquad \vartheta = -\cos(\pi u)$
$= 4\pi a \int_0^1 \sqrt{a^2 \vartheta^2 + b^2(1 - \vartheta^2)} d\vartheta$

We may confirm this integral by using the conventional formula for surfaces of revolution in $\mathbb{R}^3$. Recall from proposition 1.1.6 that $M_{(2)}$ is obtained by revolving the curve $C := \{(x, y, z) \in \mathbb{R}^3 : x = 0, y \geq 0, \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1\}$ by $2\pi$ radians about the z-axis. $C$ is then parametrized by

$$
\begin{array}{rcl}
f: \ [-\frac{\pi}{2}, \frac{\pi}{2}] & \rightarrow & \mathbb{R}^3 \\
\alpha & \mapsto & f(\alpha) = (0, a\cos\alpha, b\sin\alpha)
\end{array}
$$

The surface area of revolution for $C$ about the z-axis is

$$
\begin{aligned}
& \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\pi y \sqrt{\left(\frac{dy}{d\alpha}\right)^2 + \left(\frac{dz}{d\alpha}\right)^2} d\alpha \\
= & \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\pi a\cos\alpha \sqrt{a^2 \sin^2\alpha + b^2 \cos^2\alpha} d\alpha \\
= & \int_{-1}^{1} 2\pi a \sqrt{a^2\vartheta^2 + b^2(1-\vartheta^2)} d\vartheta \qquad \vartheta = \sin\alpha \\
= & 4\pi a \int_{0}^{1} \sqrt{a^2\vartheta^2 + b^2(1-\vartheta^2)} d\vartheta
\end{aligned}
$$

The results from both formulae agree although computation by the Riemannian structure will have to be used for a manifold which is not obtainable by revolution. Upon evaluation of the above integral we get two possible outcomes.

<u>Case 1</u>: $a > b$. In this case, area of $M_{(2)}$

$$
\begin{aligned}
& = 2\pi a \left[ \vartheta\sqrt{\vartheta^2(a^2-b^2)+b^2} + \frac{b^2}{\sqrt{a^2-b^2}} ln(\vartheta\sqrt{a^2-b^2} + \sqrt{\vartheta^2(a^2-b^2)+b^2}\ ) \right]_{\vartheta=0}^{1} \\
& = 2\pi a \left( a + \frac{b^2}{\sqrt{a^2-b^2}} ln\left(\frac{a+\sqrt{a^2-b^2}}{b}\right)\right) \\
& = 2\pi a^2 + \frac{\pi b^2}{\varepsilon} ln\left(\frac{1+\varepsilon}{1-\varepsilon}\right)
\end{aligned}
$$

<u>Case 2</u>: $b > a$. In this case, area of $M_{(2)}$

$$
\begin{aligned}
& = 2\pi a \left[ \vartheta\sqrt{b^2-\vartheta^2(b^2-a^2)} + \frac{b^2}{\sqrt{b^2-a^2}} arcsin\left(\frac{\vartheta\sqrt{b^2-a^2}}{b}\right) \right]_{\vartheta=0}^{1} \\
& = 2\pi a \left( a + \frac{b^2}{\sqrt{b^2-a^2}} arcsin\left(\frac{\sqrt{b^2-a^2}}{b}\right)\right) \\
& = 2\pi a^2 + \frac{2\pi ab}{\varepsilon} arcsin(\varepsilon)
\end{aligned}
$$

In either case, $\varepsilon$ is the eccentricity of the generating half-ellipse $C$, which is also defined to be the eccentricity of $M_{(2)}$. Case 1 is a revolution about the minor axis of $C$ which generates an OBLATE ELLIPSOID, while case 2 of revolution about the major axis of $C$ generates a PROLATE ELLIPSOID.

## 2.2 Other quantities associated to $\mathbb{R}^3$ ellipsoid derived from Riemannian structure, geodesics of $\mathbb{R}^3$ ellipsoid

### 2.2.1 The shape operator

The ellipsoid in $\mathbb{R}^3$ is of particular interest for several practical reasons. In geodetics for example, the shape of the Earth's surface is modelled as an oblate ellipsoid. $M_{(2)} \subset \mathbb{R}^3$ is the ellipsoid of least dimension for which we can discuss critical points of the arclength function within the manifold. To this effect, we will obtain standard values associated to $M_{(2)}$ derived from the Riemannian structure.

In the co-ordinate system c used in the previous chapter,
$c : [0,1] \times [0,1] \longrightarrow \mathbb{R}^3$
$(t,u) \longmapsto c(t,u) = (a\cos(2\pi t)\sin(\pi u), a\sin(2\pi t)\sin(\pi u), -b\cos(\pi u))$,
the image of a fixed line $t = t_0$ under c is called a meridian on $M_{(2)}$, while the image of a fixed line $u = u_0$ under c is called a parallel on $M_{(2)}$. Parallels and meridians on the globe are respectively lines of latitude and longitude. We compute the Gauss map N from c, where $N(c(\nu))_p$ gives the outward unit normal vector to $M_{(2)}$ at $p = c(\nu)\ \forall\ \nu \in [0,1] \times [0,1]$.

$$\frac{\partial c}{\partial t} \times \frac{\partial c}{\partial u} = (-2\pi a\sin(2\pi t)\sin(\pi u), 2\pi a\cos(2\pi t)\sin(\pi u), 0) \times (\pi a\cos(2\pi t)\cos(\pi u), \pi a\sin(2\pi t)\cos(\pi u), b\pi\sin(\pi u))$$

$$= 2\pi^2 a\sin(\pi u)(b\cos(2\pi t)\sin(\pi u), b\sin(2\pi t)\sin(\pi u), -a\cos(\pi u))$$

$$\left\| \frac{\partial c}{\partial t} \times \frac{\partial c}{\partial u} \right\| = 2\pi^2 a\sin(\pi u)\sqrt{b^2\sin^2(\pi u) + a^2\cos^2(\pi u)}$$

Hence, the Gauss map is derived;
$N : M_{(2)} \to S^2$

$$N(c(t,u))_p = \frac{(b\cos(2\pi t)\sin(\pi u), b\sin(2\pi t)\sin(\pi u), -a\cos(\pi u))}{\sqrt{b^2\sin^2(\pi u) + a^2\cos^2(\pi u)}}$$

Note that the vectors $\dfrac{\partial c}{\partial t}$ and $\dfrac{\partial c}{\partial u}$ are perpendicular from the Riemannian structure since their inner product F is 0. Hence $\dfrac{\partial c}{\partial t}|_\nu$ and $\dfrac{\partial c}{\partial u}|_\nu$ form an orthogonal basis for the tangent plane $T_{c(\nu)}M_{(2)}$ to $M_{(2)}$ at $c(\nu)$.

$$\frac{\partial^2 c}{\partial t^2} = (-4\pi^2 a\cos(2\pi t)\sin(\pi u), -4\pi^2 a\sin(2\pi t)\sin(\pi u), 0)$$
$$\frac{\partial^2 c}{\partial t\partial u} = (-2\pi^2 a\sin(2\pi t)\cos(\pi u), 2\pi^2 a\cos(2\pi t)\cos(\pi u), 0)$$
$$\frac{\partial^2 c}{\partial u^2} = (-\pi^2 a\cos(2\pi t)\sin(\pi u), -\pi^2 a\sin(2\pi t)\sin(\pi u), b\pi^2\cos(\pi u))$$
$$\langle\frac{\partial^2 c}{\partial t^2}, N\rangle = \frac{-4\pi^2 ab\sin^2(\pi u)}{\sqrt{b^2\sin^2(\pi u) + a^2\cos^2(\pi u)}} =: h_{11}$$
$$\langle\frac{\partial^2 c}{\partial t\partial u}, N\rangle = 0 =: h_{12} =: h_{21}$$
$$\langle\frac{\partial^2 c}{\partial u^2}, N\rangle = \frac{-\pi^2 ab}{\sqrt{b^2\sin^2(\pi u) + a^2\cos^2(\pi u)}} =: h_{22}$$

The matrix $[h_{ij}]$ for $1 \leq i, j \leq 2$ is called the matrix of the second fundamental form. A necessary and sufficient characterization of orientable surfaces in $\mathbb{R}^3$ is differentiability of the Gauss map. The negative differential of the Gauss map of an orientable surface is called the matrix of the shape operator of the surface. Its computation involves the matrix of the first fundamental form of the surface:

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

The matrix of the shape operator is obtained with respect to an orthogonal basis for the tangent plane at each point of the manifold.

In our case, the matrix of the shape operator $-dN$ with respect to the basis $\left(\frac{\partial c}{\partial t}, \frac{\partial c}{\partial u}\right)$ of $T_pM_{(2)}$ is given by

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$$
$$= \frac{-b}{a(b^2\sin^2(\pi u) + a^2\cos^2(\pi u))^{\frac{3}{2}}} \begin{pmatrix} b^2\sin^2(\pi u) + a^2\cos^2(\pi u) & 0 \\ 0 & a^2 \end{pmatrix}.$$

Hence we derive the following three quantities for $M_{(2)}$:

i.) The principal curvatures $\lambda_1$ and $\lambda_2$, which are the eigenvalues of the shape operator

$$\lambda_1 = \frac{-b}{a\sqrt{b^2\sin^2(\pi u) + a^2\cos^2(\pi u)}}; \quad \lambda_2 = \frac{-ab}{(b^2\sin^2(\pi u) + a^2\cos^2(\pi u))^{\frac{3}{2}}}$$

ii.) The Gaussian curvature, which is the product of the two eigenvalues of the shape operator

$$\lambda_1\lambda_2 = \frac{b^2}{\left(b^2\sin^2(\pi u) + a^2\cos^2(\pi u)\right)^2}$$

iii.) The mean curvature, which is the mean of the two eigenvalues of the shape operator

$$\frac{1}{2}(\lambda_1 + \lambda_2) = \frac{-(a^2b + a^2b\cos^2(\pi u) + b^3\sin^2(\pi u))}{2a(b^2\sin^2(\pi u) + a^2\cos^2(\pi u))^{\frac{3}{2}}}$$

Notably, from the preceding derivations, the Gauss map is a diffeomorphism between the unit sphere $S^2$ and $M_{(2)}$. We can define other such diffeomorphisms, two of which are given below.

1) $f_1 : S^2 \to M_{(2)}$
$(x, y, z) \longmapsto T(x, y, z) = (ax, ay, bz)$

2) $f_2 : S^2 \to M_{(2)}$
$$(x, y, z) \longmapsto \frac{(abx, aby, abz)}{\sqrt{b^2(1 - z^2) + a^2z^2}}$$

The map $f_1$ is linear while $f_2$ radially maps each point of the unit sphere to the proximate point of $M_{(2)}$ which is collinear with the sphere's point and the origin.

### 2.2.2 Geodesics of the 2-dimensional ellipsoid

We will now present vital facts in connection with geodesics on the ellipsoid $M_{(2)}$. Any regular curve $\alpha$ drawn on a regular surface $M \subset \mathbb{R}^3$ is characterized by the equation $\alpha''(s) = k_g(s)\mathbf{n} + k_N(s)\mathbf{N}_M$ where $\alpha$ is parametrized by arclength := s. The variables $k_g(s), k_N(s), \mathbf{N}_M$ and $\mathbf{n}$ are respectively the geodesic curvature of $\alpha$, normal curvature of $\alpha$, Gauss map or outward unit normal to M, and a unit normal to the curve $\alpha$ lying on the tangent plane to M. The curve $\alpha$ is a geodesic of M iff its geodesic curvature vanishes everywhere. Hence, $\alpha''(s) = k_N(s)\mathbf{N}_M$ for a geodesic $\alpha \subset M$.

Because $\alpha$ is parametrized by arclength, $\alpha''(s) = k\mathbf{N}_\alpha$, where $k$ is the curvature of $\alpha$ and $\mathbf{N}_\alpha$ is the principal unit normal vector to $\alpha$. Hence, for a geodesic curve $\alpha \subset M$, we have $k\mathbf{N}_\alpha = k_N\mathbf{N}_M$. This implies that the principal unit normal to a geodesic curve at any point coincides with the outward unit normal to M at that point. Minimizers of the arclength function on M are smooth curves called minimal geodesics. In $\mathbb{R}^3$, the arclength function is given by

$$I = \int ds = \int \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$$

where $ds$ is the element of arclength one-form, which represents an infinitesimal increment in the arclength function. We have $x = x(t,u), y = y(t,u), z = z(t,u)$ so that $dx = \frac{\partial x}{\partial t}dt + \frac{\partial x}{\partial u}du$ and
$(dx)^2 = \frac{\partial x}{\partial t}^2 (dt)^2 + 2\frac{\partial x}{\partial t}\frac{\partial x}{\partial u}dtdu + \frac{\partial x}{\partial u}^2 (du)^2$.
Obtaining similar expansions for $(dy)^2$ and $(dz)^2$, we get $I =$

$$\int \sqrt{[\frac{\partial x}{\partial t}^2 + \frac{\partial y}{\partial t}^2 + \frac{\partial z}{\partial t}^2](dt)^2 + 2[\frac{\partial x}{\partial t}\frac{\partial x}{\partial u} + \frac{\partial y}{\partial t}\frac{\partial y}{\partial u} + \frac{\partial z}{\partial t}\frac{\partial z}{\partial u}]dtdu + [\frac{\partial x}{\partial u}^2 + \frac{\partial y}{\partial u}^2 + \frac{\partial z}{\partial u}^2](du)^2}$$

It is clear that the coefficients in this expression constitute the Riemannian structure;

$$\frac{\partial x}{\partial t}^2 + \frac{\partial y}{\partial t}^2 + \frac{\partial z}{\partial t}^2 = \left\|\frac{\partial f}{\partial t}\right\|^2 = E$$
$$\frac{\partial x}{\partial t}\frac{\partial x}{\partial u} + \frac{\partial y}{\partial t}\frac{\partial y}{\partial u} + \frac{\partial z}{\partial t}\frac{\partial z}{\partial u} = \langle\frac{\partial f}{\partial t}, \frac{\partial f}{\partial u}\rangle = F$$
$$\frac{\partial x}{\partial u}^2 + \frac{\partial y}{\partial u}^2 + \frac{\partial z}{\partial u}^2 = \left\|\frac{\partial f}{\partial u}\right\|^2 = G,$$

where $f$ is the parametrization,

$$\begin{array}{rccc} f: & V \subseteq \mathbb{R}^2 & \longrightarrow & M \subseteq \mathbb{R}^3 \\ & (t,u) & \longmapsto & (x,y,z) \end{array}$$

Now, let $\dfrac{du}{dt} = u', \dfrac{dt}{du} = t'$, to re-express $I$.

$$I = \int \sqrt{E + 2Fu' + G(u')^2}dt,$$

or equivalently

$$I = \int \sqrt{E(t')^2 + 2Ft' + G}du.$$

Let $L = \sqrt{E(t')^2 + 2Ft' + G}$ so that $I = \int Ldu$

$$\frac{\partial L}{\partial t} = \frac{1}{2}(E(t')^2 + 2Ft' + G)^{\frac{-1}{2}}(\frac{\partial E}{\partial t}(t')^2 + 2\frac{\partial F}{\partial t}t' + \frac{\partial G}{\partial t})$$
$$\frac{\partial L}{\partial t'} = \frac{1}{2}(E(t')^2 + 2Ft' + G)^{\frac{-1}{2}}(2E(t') + 2F)$$

We may now directly apply the Euler - Lagrange differential equation which minimizes functionals of the type $I$. Hence, we must have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial u'}\right) = \frac{\partial L}{\partial u} \text{ and } \frac{d}{du}\left(\frac{\partial L}{\partial t'}\right) = \frac{\partial L}{\partial t}.$$

Since none of the terms from the Riemannian structure depend on t in the case of our parametrization for $M_{(2)}$, the second equation will yield a clearer result as taking the partial derivative with respect to t (on the right) will yield zero. We then have a geodesic equation given as

$$\frac{d}{du}\left(\frac{(E(t') + F)}{\sqrt{E(t')^2 + 2Ft' + G}}\right)$$
$$= \frac{1}{2}(E(t')^2 + 2Ft' + G)^{\frac{-1}{2}}(\frac{\partial E}{\partial t}(t')^2 + 2\frac{\partial F}{\partial t}t' + \frac{\partial G}{\partial t}) = 0$$
$$\Longrightarrow E(t') + F = \gamma\sqrt{E(t')^2 + 2Ft' + G} \qquad \text{for a constant } \gamma$$
$$\Longrightarrow t' = \frac{dt}{du} = \sqrt{\frac{\gamma^2 G}{E(E - \gamma^2)}} \ldots eq(2.1) \text{ since F is zero.}$$

Using the Riemannian structure obtained from the parametrization c, we get

$$\frac{dt}{du} = \sqrt{\frac{\gamma^2(a^2\cos^2(\pi u) + b^2\sin^2(\pi u))}{4a^2\sin^2(\pi u)(4\pi^2a^2\sin^2(\pi u) - \gamma^2)}}$$

$$\Rightarrow t = \frac{\gamma}{2a}\int\sqrt{\frac{a^2\cos^2(\pi u) + b^2\sin^2(\pi u)}{\sin^2(\pi u)(4\pi^2a^2\sin^2(\pi u) - \gamma^2)}}du$$

The appearance of each geodesic curve depends largely on the constant $\gamma$, which may either be determined by the starting point and azimuth or by starting and ending points of the curve. The azimuth is simply the angle between the path and meridian measured clockwise from the meridian on the tangent plane at the starting point. Assume we are given starting point p and azimuth $\beta$, we can always obtain the corresponding vector $v_p \in T_pM_{(2)}$ which under the used parametrization (c) has a pre-image, say $\psi_\nu \in T_\nu(]0,1[^2)$, as long as $\nu$ is an interior point of $([0,1] \times [0,1])$. To be more precise, we have $c_*\psi_\nu = v_p$, whenever $\nu \in ]0,1[\times]0,1[$. Note that the only points of $M_{(2)}$ whose pre-images under c do not belong to $]0,1[\times]0,1[$ constitute the meridian $t = 0 \equiv 1$. If the starting point belongs to the meridian t = 0 (which is a $\mu$ - null subset of $M_{(2)}$) , then computations are easily improvised. Obtaining the values of $dt(\psi_\nu) := \psi_1$ and $du(\psi_\nu) := \psi_2$ we have that $t'|_\nu = \dfrac{\psi_1}{\psi_2}$. Substituting this value in our geodesic equation (2.1) immediately yields $\gamma$, which here is unique.

On the other hand, if we are given the starting and ending points of the geodesic curve then we use Clairaut's geodesic equation, which is applicable to 2-dimensional manifolds of revolution in $\mathbb{R}^3$. Clairaut's equation states $r\sin(\beta) = constant$, where $r$ is the radius of the parallel of latitude given by $\dfrac{a\cos(\pi u)}{(1-\varepsilon^2\sin^2(\pi u))^{\frac{1}{2}}}$, $\beta$ is the azimuth, $\varepsilon$ is the eccentricity of $M_{(2)}$ and $a$ is the radius of the equator. It must be noted that in this latter case, the value of $\gamma$ obtained might not be unique, hence nullifying the uniqueness of geodesic curves given starting and ending points.

In the geodesic equation (2.1), we may express t in terms of the eccentricity of $M_{(2)}$ for the oblate and prolate cases.

If $M_{(2)}$ is oblate, then $a > b$ and $1 - \varepsilon^2 = \dfrac{b^2}{a^2}$ in which case

$$t = \frac{\gamma}{2} \int \sqrt{\frac{\cos^2(\pi u) + (1-\varepsilon^2)\sin^2(\pi u)}{\sin^2(\pi u)(4\pi^2 a^2 \sin^2(\pi u) - \gamma^2)}} du$$

If $M_{(2)}$ is prolate then $b > a$ and $1 - \varepsilon^2 = \frac{a^2}{b^2}$ in which case

$$t = \frac{\gamma}{2} \int \sqrt{\frac{(1-\varepsilon^2)\cos^2(\pi u) + \sin^2(\pi u)}{\sin^2(\pi u)(4\pi^2 a^2 \sin^2(\pi u) - \gamma^2)}} du$$

The antiderivates of the above integrands for ellipsoidal geodesics cannot be expressed in terms of elementary functions. In any event, the meridians and equator of $M_{(2)}$ can be computationally proven to be geodesics.
For a meridian $t = t_0$ fixed, we have $\frac{dt}{du} = 0$ making the constant $\gamma$ to be 0 in our derived geodesic equation.

We will now also confirm that for a meridian $\alpha = c(t_0, u)$, the outward unit normal to the manifold and the principal unit normal to the meridian as a curve (resp. $\mathbf{N}_M$ and $\mathbf{N}_\alpha$) coincide.

For any regular curve $\alpha$ in $\mathbb{R}^3$, we have
$\kappa \mathbf{N}_\alpha = \alpha''(s) = T'(s) = \frac{dT}{du}\frac{du}{ds}$,
where $T = \frac{\alpha'(u)}{\|\alpha'(u)\|}$ is the unit tangent to $\alpha, \kappa$ its curvature and $s$ its arclength.
$\alpha = (a\delta \sin(\pi u), a\zeta \sin(\pi u), -b\cos(\pi u))$ where $\delta = \cos(2\pi t_0), \zeta = \sin(2\pi t_0) = \pm\sqrt{1-\delta^2}$. As such,

$$\alpha'(u) = (a\pi\delta \cos(\pi u), a\pi\zeta \cos(\pi u), b\pi \sin(\pi u))$$

$$\|\alpha'(u)\| = \sqrt{a^2\pi^2\cos^2(\pi u) + b^2\pi^2\sin^2(\pi u)}$$

$$s(u) = \int_{u_0}^{u} \|\alpha'(v)\| dv$$

$$= \int_{u_0}^{u} \sqrt{a^2\pi^2\cos^2(\pi v) + b^2\pi^2\sin^2(\pi v)} dv$$

$$\frac{ds}{du} = \pi\sqrt{a^2\cos^2(\pi u) + b^2\sin^2(\pi u)}$$

$$T = \frac{(a\pi\delta \cos(\pi u), a\pi\zeta \cos(\pi u), b\pi \sin(\pi u))}{\pi\sqrt{a^2\cos^2(\pi u) + b^2\sin^2(\pi u)}}$$

$$\frac{dT}{du} = \frac{(-\pi b^2 a\delta \sin(\pi u), -\pi b^2 a\zeta \sin(\pi u), \pi b a^2 \cos(\pi u))}{(a^2\cos^2(\pi u) + b^2\sin^2(\pi u))^{\frac{3}{2}}}$$

$$\frac{dT}{du}\frac{du}{ds} = \frac{(-b^2 a\delta\sin(\pi u), -b^2 a\zeta\sin(\pi u), ba^2\cos(\pi u))}{\left(a^2\cos^2(\pi u) + b^2\sin^2(\pi u)\right)^2} = \lambda_2 \mathbf{N}_M.$$

Recall $\lambda_2$ to be one of the principal curvatures of $M_{(2)}$.

We similarly confirm the coincidence of $\mathbf{N}_M$ and $\mathbf{N}_\phi$ for the equator $\phi$,

$$\phi = c(t, \tfrac{1}{2}) = (a\cos(2\pi t), a\sin(2\pi t), 0)$$

$$\phi'(t) = (-2\pi a\sin(2\pi t), 2\pi a\cos(2\pi t), 0)$$

$$\|\phi'(t)\| = 2\pi a$$

$$s(t) = \int_{t_0}^{t} 2\pi a dv = 2\pi at - 2\pi at_0$$

$$\frac{ds}{dt} = 2\pi a$$

$$T = (-\sin(2\pi t), \cos(2\pi t), 0)$$

$$\kappa \mathbf{N}_\phi = \frac{dT}{dt}\frac{dt}{ds}$$

$$= (-2\pi\cos(2\pi t), -2\pi\sin(2\pi t), 0)\frac{1}{2\pi a}$$

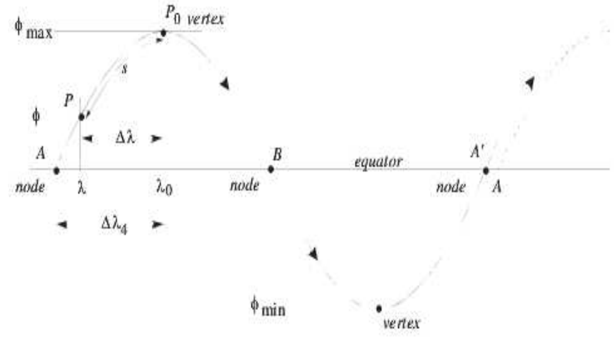
$$= (\tfrac{-1}{a}\cos(2\pi t), \tfrac{-1}{a}\sin(2\pi t), 0)$$

The Gauss map evaluated on the equator, $\mathbf{N}_M = (\cos(2\pi t), \sin(2\pi t), 0)$ and the corresponding prinicpal curvature, $\lambda_1 = \dfrac{-1}{a}$.

Hence, we observe that $\kappa \mathbf{N}_\phi = \lambda_1 \mathbf{N}_M$.

Other geodesic curves besides meridians and the equator appear in particular patterns. For instance, in the case of the oblate ellipsoid which has been extensively analyzed, geodesics appear as periodic curves oscillating about the equator. They do not repeat after a complete revolution.

As a physical application, geodesics are the only curves on a surface along which a particle can move without accelerating tangentially. For this reason, they are also recognized as kinetic energy minimizers.




***Schematic of oscillation of a geodesic on an oblate ellipsoid.***

## 2.3 Manifolds in higher dimensions: volume element, geodesics

In theory, differential forms on manifolds in higher dimensional Euclidean spaces are useful to extrapolate results from lower dimensions. We illustrate this in the following analysis of the ellipsoid $M_{(n)}$ for $n \geq 3$. Note that axial symmetry and uniqueness of the outward unit normal at each point on the manifold are key in obtaining results in computations and proofs. A hypersurface in $\mathbb{R}^{n+1}$ is an n-dimensional submanifold of the space, giving us that $M_{(n)}$ is a hypersurface in $\mathbb{R}^{n+1}$. We now proceed to examine hypersufaces' volume forms and their geodesics, which fundamentally involve the element of arclength.

### 2.3.1 Higher dimensional volume forms

**Theorem 2.3.1.** *Let $M$ be a hypersurface in $\mathbb{R}^{n+1}$ and*

$$\begin{array}{rccl} f: & U \subseteq \mathbb{R}^n & \longrightarrow & M \\ & u = (u_1, u_2, ..., u_n) & \longmapsto & f(u) = p \end{array}$$

*be a differentiable parametrization of $M$ then the volume of $M$ can be obtained from the formula*

$$vol(M) = \int_U \sqrt{det[g_{ij}]}$$

*where $g_{ij}$ is from the Riemannian structure;*

$$g_{ij} = g_{ji} = \langle \frac{\partial f}{\partial u_i}, \frac{\partial f}{\partial u_j} \rangle \text{ for } 1 \leq i, j \leq n,$$

*of the Riemannian metric of $M$ defined by*

$$g = \sum_{i,j=1}^{n} g_{ij} du_i \otimes du_j.$$

*Proof.* The aim is to get $\int_M dV$ as an equal integral having U as the domain of integration using the pullback $f^*$.

$$\int_M dV = \int_U f^*(dV)$$

Define $\omega \in \bigwedge^n(T_pM)$ by

$$\omega(v_1, \cdots, v_n) = det \begin{pmatrix} n(p) \\ v_1 \\ \vdots \\ v_n \end{pmatrix}$$

where n(p) is the outward unit normal to $M$ at p. If $(v_1, \cdots, v_n)$ form an orthonormal basis for $T_pM$ with $[v_1, \cdots, v_n] = \mu_p$ then $\omega(v_1, \cdots, v_n) = 1$ meaning that $\omega$ is the volume element dV. We may define

$$\omega(v_1, \cdots, v_n) := \langle (v_1 \times \cdots \times v_n), n(p) \rangle = \pm \| v_1 \times \cdots \times v_n \|$$

since the unit vector n(p) is perpendicular to each of the vectors $v_1, \cdots, v_n$. The operation above is a multiple cross product which can be executed with n vectors in $\mathbb{R}^{n+1}$ to produce a mutually orthogonal vector. As we will see briefly, this operation induces an inner product on $\mathbb{R}^n$ when the norm is applied to it.

For each $u \in \overset{\circ}{U}$, we have

$$\begin{aligned} f^*(dV)((e_1)_u, \cdots, (e_n)_u) &= dV(f_*((e_1)_u), \cdots, f_*((e_n)_u)) \\ &= dV\left(\frac{\partial f}{\partial u_1}, \cdots, \frac{\partial f}{\partial u_n}\right) \\ &= \omega\left(\frac{\partial f}{\partial u_1}, \cdots, \frac{\partial f}{\partial u_n}\right) \end{aligned}$$

Let $(v_1, \cdots, v_n)$ be an orthonormal basis for $T_pM$, then we have $\omega(v_1, \cdots, v_n) = \pm 1$. We can then express each of the vectors $\frac{\partial f}{\partial u_i}$ as a linear combination of the basis vectors. Let us set

$$\begin{pmatrix} \frac{\partial f}{\partial u_1} \\ \vdots \\ \frac{\partial f}{\partial u_n} \end{pmatrix} = [a_{ij}] \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

so that $\frac{\partial f}{\partial u_i} = \sum_{k=1}^n a_{ik} v_k$

and $\langle \frac{\partial f}{\partial u_i}, \frac{\partial f}{\partial u_j} \rangle = \sum_{k,l=1}^n a_{ik} a_{jl} \langle v_k, v_l \rangle = \sum_{k=1}^n a_{ik} a_{jk}$

But from theorem 1.2.1, we have $\omega\left(\frac{\partial f}{\partial u_1},\cdots,\frac{\partial f}{\partial u_n}\right) = \pm det[a_{ij}]$ which implies that

$$\begin{aligned}\left(\omega(\frac{\partial f}{\partial u_1},\cdots,\frac{\partial f}{\partial u_n})\right)^2 &= (det[a_{ij}])^2 \\ &= (det[a_{ij}])(det[a_{ji}]) \\ &= det([a_{ij}][a_{ji}]) \\ &= det[\zeta_{ij}]\end{aligned}$$

where $\zeta_{ij} = \sum_{k=1}^{n} a_{ik}a_{jk}$ which we have already seen to equal

$$\langle\frac{\partial f}{\partial u_i},\frac{\partial f}{\partial u_j}\rangle := g_{ij}.$$

Hence,

$$\left|\omega\left(\frac{\partial f}{\partial u_1},\cdots,\frac{\partial f}{\partial u_n}\right)\right| = \left\|\frac{\partial f}{\partial u_1}\times\cdots\times\frac{\partial f}{\partial u_n}\right\| = \sqrt{det[g_{ij}]}$$

and this gives us

$$\begin{aligned}vol(M) &= |\int_M dV| \\ &= |\int_U f^*dV((e_1),\cdots,(e_n))|du_1\wedge du_2\wedge\cdots\wedge du_n \\ &= \int_U \sqrt{det[g_{ij}]}du_1\wedge du_2\wedge\cdots\wedge du_n\end{aligned}$$

□

Of course, the system of coordinates in U must cover the manifold exactly once to obtain the correct volume, meaning that the parametrization $f: U \to M$ should be a bijection, except possibly for a negligible subset of M.

***Remark*** - Observe that we may express $g_u$ as $g_u(v,w) = v^T[g_{ij}]w$ for $v, w \in \mathbb{R}^n_u$ which is bilinear, symmetric and positive definite and so is an inner product on the vector space $\mathbb{R}^n_u$. Concretely, we have $g_p$ as the usual inner product on $T_pM$ pulled back to $\mathbb{R}^n_u$ by the parametrization $f$ since we have $v^T[g_{ij}]w = \langle Df(v), Df(w)\rangle$. The matrix $[g_{ij}]$ is also referred to as the matrix of the first fundamental form.

The matrix $[h_{ij}]$ for $1 \leq i, j \leq n$ is the second fundamental form where

$$h_{ij} = \langle \frac{\partial^2 f}{\partial u_i \partial u_j}, \psi \rangle$$

and $\psi$ is the outward unit normal to the hypersurface.

The matrix of the shape operator of $M$ is given by $[g_{ij}]^{-1}[h_{ij}]$ and we obtain the following quantities from it:

i.) The principal curvatures $\lambda_1, \cdots, \lambda_n$, which are the eigenvalues of the shape operator;

ii.) The Gauss - Kronecker curvature $\lambda_1 \cdots \lambda_n$, which is the product of the $n$ eigenvalues of the shape operator and also equals its determinant;

iii.) The mean curvature $\frac{1}{n}(\lambda_1 + \cdots + \lambda_n)$, which is the mean of the $n$ eigenvalues of the shape operator and also equals its trace divided by $n$.

We also have a matrix of the third fundamental form of $M$ given by

$$\left[ \langle \frac{\partial \psi}{\partial u_i}, \frac{\partial \psi}{\partial u_j} \rangle \right]_{ij},$$

where $\psi$ is the outward unit normal to the hypersurface.

The next theorem gives a means of evaluating volumes of n-dimensional submanifolds of $\mathbb{R}^n$ from relevant transformations or maps.

**Theorem 2.3.2.**
*Let $S$ and $P$ be differentiable n-dimensional submanifolds of $\mathbb{R}^n$ with $P = f(S)$ for a smooth map*

$$\begin{array}{cccc} f: & S & \longrightarrow & \mathbb{R}^n \\ & (u_1, u_2, ..., u_n) & \longmapsto & (x_1, x_2, ..., x_n) \end{array}$$

*Then the volume of $P$ is given by*

$$\int_S df^1 \wedge df^2 \wedge \cdots \wedge df^n = \int_S J(f) du_1 \wedge du_2 \wedge \cdots \wedge du_n.$$

*Proof.* We have by definition

$$\int_P \omega = \int_S f^*\omega$$

where $\omega$ is the volume element of P. In this case, the volume element for S, P and $\mathbb{R}^n$ is the determinant given by $\omega = det = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ on the range and $det = du_1 \wedge du_2 \wedge \cdots \wedge du_n$ on the domain. Hence the volume of P equals

$$\int_S f^*(dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n)$$
$$= \int_S f^*dx_1 \wedge f^*dx_2 \wedge \cdots \wedge f^*dx_n$$
$$= \int_S df^1 \wedge df^2 \wedge \cdots \wedge df^n$$

These equalities follow from the properties of the pullback operator listed in section 1.2.2. Also, we derive the volume formula as follows.

$$\int_S f^*\omega(e_1, \cdots, e_n) du_1 \wedge du_2 \wedge \cdots \wedge du_n$$
$$= \int_S \omega(f)(f_*(e_1), \cdots, f_*(e_n)) du_1 \wedge du_2 \wedge \cdots \wedge du_n$$
$$= \int_S det(f_*(e_1), \cdots, f_*(e_n)) du_1 \wedge du_2 \wedge \cdots \wedge du_n$$
$$= \int_S det(\frac{\partial f}{\partial u_1}, \cdots, \frac{\partial f}{\partial u_n}) du_1 \wedge du_2 \wedge \cdots \wedge du_n$$
$$= \int_S J(f) du_1 \wedge du_2 \wedge \cdots \wedge du_n \qquad (J(f) \text{ is the Jacobian determinant})$$ □

### 2.3.2 Higher dimensional geodesics

The calculus of variations in higher dimensions is key to studying minimal geodesics within the hypersuface $M$, which in any case are locally length-minimizing curves. $M$ as a subset of $\mathbb{R}^{n+1}$ is parametrized by n variables so we set as a parametrization for $M$

$$f : V \subseteq \mathbb{R}^n \longrightarrow M \subset \mathbb{R}^{n+1} \quad ; (u_1, u_2, \cdots, u_n) \longmapsto (x_1, x_2, \cdots, x_{n+1})$$

The arclength function is given by

$$I = \int ds = \int \sqrt{(dx_1)^2 + (dx_2)^2 + \cdots + (dx_{n+1})^2}$$

And

$$dx_i = \sum_{j=1}^{n} \frac{\partial x_i}{\partial u_j} du_j$$

which gives us that

$$(dx_i)^2 = \sum_{j=1}^{n} (\frac{\partial x_i}{\partial u_j})^2 (du_j)^2 + 2 \sum_{\substack{k,l=1 \\ k \neq l}}^{n} \frac{\partial x_i}{\partial u_k} \frac{\partial x_i}{\partial u_l} du_k du_l$$

We then have

$$I = \int \sqrt{\sum_{j=1}^{n} \left[ \sum_{i=1}^{n+1} (\frac{\partial x_i}{\partial u_j})^2 (du_j)^2 \right] + 2 \sum_{\substack{k,l=1 \\ k \neq l}}^{n} \left[ \sum_{i=1}^{n+1} \frac{\partial x_i}{\partial u_k} \frac{\partial x_i}{\partial u_l} du_k du_l \right]}$$

The expression under the integral is the element of arclength for $M$ while the coefficients therein constitute the Riemannian structure. More specifically, the coefficient of $du_i du_j$ is $\langle \frac{\partial f}{\partial u_i}, \frac{\partial f}{\partial u_j} \rangle$. We may then write,

$$I = \int \sqrt{\sum\nolimits_{i,j=1}^{n} g_{ij} du_i du_j}.$$

Now, we illustrate computations using the specific manifold $M_{(n)}$. Let us consider the parametrization for $M_{(n)}$ given by

$f_0 : [\frac{-\pi}{2}, \frac{\pi}{2}]^{n-1} \times [0, 2\pi] \longrightarrow M_{(n)} \;\; ; (u_1, u_2, \cdots, u_n) \longmapsto (x_1, x_2, \cdots, x_{n+1});$
$f_0^1 := x_1 = a \prod\limits_{j=1}^{n} \cos u_j$
$f_0^i := x_i = a \prod\limits_{j=1}^{n-i+1} \cos u_j \sin u_{n-i+2}, \quad$ if $2 \leq i \leq n$
$f_0^{n+1} := x_{n+1} = b \sin u_1$

The parametrization given above covers the manifold exactly once and it yields the following system of partial derivatives.

$$\frac{\partial x_i}{\partial u_j} = \begin{cases} 0 & \text{for } i \geq n-j+3. \textit{ Otherwise,} \\ -a \sin u_j \prod\limits_{k=1, k \neq j}^{n} \cos u_k & \text{for } i = 1 \\ -a \sin u_j (\prod\limits_{k=1, k \neq j}^{n-i+1} \cos u_k) \sin u_{n-i+2} & \text{for } 2 \leq i \leq n-j+1 \\ a \prod\limits_{k=1}^{n-i+2} \cos u_k & \text{for } i = n-j+2 \neq n+1 \\ b \cos u_1 & \text{for } i = n+1 \end{cases}$$

When $j < l < n$, then

$$\langle \frac{\partial f_0}{\partial u_j}, \frac{\partial f_0}{\partial u_l} \rangle = \sum_{i=1}^{n+1} \frac{\partial x_i}{\partial u_j} \frac{\partial x_i}{\partial u_l}$$

$= a^2 \sin u_j \sin u_l \cos u_j \cos u_l (\prod_{k=1,k\neq j,l}^{n} \cos^2 u_k + \sin^2 u_n \prod_{k=1,k\neq j,l}^{n-1} \cos^2 u_k + \sin^2 u_{n-1} \prod_{k=1,k\neq j,l}^{n-2} \cos^2 u_k +$
$\ldots + \sin^2 u_{l+1} \prod_{k=1,k\neq j,l}^{l} \cos^2 u_k) - a^2 \sin u_j \sin u_l \cos u_j \cos u_l (\prod_{k=1,k\neq j,l}^{l-1} \cos^2 u_k)$
$= a^2 \sin u_j \sin u_l \cos u_j \cos u_l (\prod_{k=1,k\neq j,l}^{l-1} \cos^2 u_k)(\prod_{k=l+1}^{n} \cos^2 u_k + \sin^2 u_n \prod_{k=l+1}^{n-1} \cos^2 u_k +$
$\sin^2 u_{n-1} \prod_{k=l+1}^{n-2} \cos^2 u_k + \ldots + \sin^2 u_{l+1} - 1) \quad = \quad 0.$

We can similarly check that $\langle \frac{\partial f_0}{\partial u_i}, \frac{\partial f_0}{\partial u_n} \rangle = 0$ for $1 \leq i \leq n-1$, hence confirming that $\left( \frac{\partial f_0}{\partial u_i} \right)_{1 \leq i \leq n}$ forms an orthogonal basis for $T_p M_{(n)}$. Using the parametrization $f_0$, we conveniently have an elimination of all terms $g_{ij}$ from the expression $ds$ whenever $i \neq j$. Our arclength function then becomes

$$I = \int \sqrt{\sum_{i=1}^{n} g_{ii} (du_i)^2} = \int \sqrt{g_{jj} + \sum_{\substack{i=1 \\ i \neq j}}^{n} g_{ii} \left( \frac{du_i}{du_j} \right)^2} du_j$$

Geodesics are the curves on $M_{(n)}$ for which their principal unit normals everywhere coincide with the outward unit normal to the manifold. Let the problem posed be to find a geodesic path within the manifold $M_{(n)}$ given a starting point and direction, in other words, $v_p \in T_p M_{(n)}$, then the geodesic is unique and we may proceed by use of Euler-Lagrange differential equations. In executing this process, we must make pairwise selections of the $u_i$'s to vary. By repeatedly varying a certain one along with one of the others at a time, we create a system of $n-1$ differential equations in $n$ unknowns. Such a system would yield the parametric equation of a curve in $\mathbb{R}^{n+1}$ when restricted to $M_{(n)}$, if it could be reduced. Observing that

$$g_{jj} = \begin{cases} a^2 \sin^2 u_1 + b^2 \cos^2 u_1 & \text{for } j = 1 \\ (a^2) \prod_{k=1}^{j-1} \cos^2 u_k & \text{for } 2 \leq j \leq n \end{cases}$$

we see that none of the terms from the Riemannian structure depend on $u_n$ so we will begin by varying this term while deriving the Euler-Lagrange equations.

$I = \int \sqrt{g_{11} + \sum_{j=2}^{n} g_{jj} \left(\frac{du_j}{du_1}\right)^2} du_1$ and we get the following as an Euler-Lagrange equation when minimizing I.

$\frac{\partial L_1}{\partial u_n} - \frac{d}{du_1}\left(\frac{\partial L_1}{\partial u'_n}\right) = 0$, where

$L_1 = \sqrt{g_{11} + \sum_{j=2}^{n} g_{jj} \left(\frac{du_j}{du_1}\right)^2}$ and $u'_n = \frac{du_n}{du_1}$

$$\begin{aligned} \frac{\partial L_1}{\partial u'_n} &= \frac{g_{nn} u'_n}{L_1} \\ &= \frac{(a^2) \prod_{k=1}^{n-1} \cos^2 u_k \frac{du_n}{du_1}}{\sqrt{g_{11} + \sum_{j=2}^{n} g_{jj} \left(\frac{du_j}{du_1}\right)^2}} \end{aligned}$$

Since $\frac{\partial L_1}{\partial u_n} = 0$ , then $\frac{\partial L_1}{\partial u'_n}$ is a constant. Therefore,

$(a^2) \prod_{k=1}^{n-1} \cos^2 u_k \frac{du_n}{du_1} = \gamma \sqrt{g_{11} + \sum_{j=2}^{n} g_{jj} \left(\frac{du_j}{du_1}\right)^2} \ldots eq(2.2)$

for a real constant $\gamma$.

Likewise, we obtain similar geodesic differential equations by repeatedly varying $u_n$ and the other $u_i$'s one at a time. We can preview each equation to observe the appearance of the geodesic solution in submanifolds of $M_{(n)}$. In the equation just obtained, when we set each $u_i$ to 0 except $u_1$ and $u_n$, we view the appearance of the geodesic curve in $(\mathbb{R} \times \mathbb{R} \times \{0\}^{n-2} \times \mathbb{R}) \cap M_{(n)}$, adhering to the axial orientation specified in the parametrization $f_0$. The co-ordinates in this submanifold are $(a \cos u_1 \cos u_n, a \cos u_1 \sin u_n, \overbrace{0, 0, \cdots, 0}^{n-2}, b \sin u_1)$, which is clearly isometric to $M_{(2)}$. Our equation then becomes

$a^2 \cos^2 u_1 \left(\frac{du_n}{du_1}\right) = \gamma \sqrt{a^2 \sin^2 u_1 + b^2 \cos^2 u_1 + a^2 \cos^2 u_1 \left(\frac{du_n}{du_1}\right)^2}$

$\implies \frac{du_n}{du_1} = \gamma \sqrt{\frac{a^2 \sin^2 u_1 + b^2 \cos^2 u_1}{a^2 \cos^2 u_1 (a^2 \cos^2 u_1 - \gamma^2)}}$

The associated solution here has identical properties to what we have for geodesics of $M_{(2)}$. (This becomes clear when we apply the parametrization

$\Phi$ from section 1 of this chapter in the geodesic equation (2.1) derived afterwards in section 2.)

Now, we vary $u_n$ and $u_k; 1 \neq k \neq n$ following the above process.

$$I = \int \sqrt{g_{kk} + \sum_{\substack{j=1 \\ j\neq k}}^{n} g_{jj} \left(\frac{du_j}{du_k}\right)^2} du_k$$

$$\frac{\partial L_k}{\partial u_n} - \frac{d}{du_k}\left(\frac{\partial L_k}{\partial u'_n}\right) = 0, \text{ where}$$

$$L_k = \sqrt{g_{kk} + \sum_{\substack{j=1 \\ j\neq k}}^{n} g_{jj} (\frac{du_j}{du_k})^2} \text{ and } u'_n = \frac{du_n}{du_k}$$

$$\Longrightarrow \frac{\partial L_k}{\partial u'_n} = \frac{g_{nn} u'_n}{L_k} = \gamma \text{ } (\gamma \text{ is a real constant}).$$

Hence, we have

$$g_{nn}\frac{du_n}{du_k} = \gamma \sqrt{g_{kk} + \sum_{\substack{j=1 \\ j\neq k}}^{n} g_{jj} \left(\frac{du_j}{du_k}\right)^2} \ldots eq(2.3)$$

Setting each $u_i$ to 0 except $u_k$ and $u_n$, we view the appearance of the geodesic curve in $(\mathbb{R}\times\mathbb{R}\times\{0\}^{n-k-1}\times\mathbb{R}\times\{0\}^{k-1})\cap M_{(n)}$, adhering to the axial orientation specified in the parametrization $f_0$. The co-ordinates in this submanifold are $(a\cos u_k \cos u_n, a\cos u_k \sin u_n, \overbrace{0, 0, \cdots, 0}^{n-k-1}, a\sin u_k, \overbrace{0, 0, \cdots, 0}^{k-1})$ which is isometric to $aS^2$. Our equation then becomes

$$a^2\cos^2 u_k \left(\frac{du_n}{du_k}\right) = \gamma \sqrt{a^2 + a^2\cos^2 u_k \left(\frac{du_n}{du_k}\right)^2}$$

$$\Rightarrow \frac{du_n}{du_k} = \gamma \sqrt{\frac{1}{\cos^2 u_k (a^2\cos^2 u_k - \gamma^2)}}$$

$$\Rightarrow u_n = \gamma \int \sqrt{\frac{1}{\cos^2 u_k (a^2\cos^2 u_k - \gamma^2)}} du_k$$

$$= \int \frac{du_k}{\cos u_k \sqrt{(\frac{a}{\gamma})^2 \cos^2 u_k - 1}}$$

$$= Arctan\left[\frac{\sin u_k}{\sqrt{(\frac{a}{\gamma})^2\cos^2 u_k - 1}}\right] + \delta \qquad (\delta \in \mathbb{R} \text{ is constant})$$

$$= Arcsin\left[\frac{Tanu_k}{\sqrt{(\frac{a}{\gamma})^2 - 1}}\right] + \delta$$

$$\Rightarrow \sin(u_n - \delta) = \frac{Tanu_k}{\sqrt{(\frac{a}{\gamma})^2 - 1}}$$

$$\Rightarrow \sin u_n \cos\delta - \cos u_n \sin\delta = \frac{1}{\cos u_k}\frac{\sin u_k}{\sqrt{(\frac{a}{\gamma})^2 - 1}}$$

$$\Rightarrow a\cos u_k \sin u_n \cos\delta - a\cos u_k \cos u_n \sin\delta = \frac{a\sin u_k}{\sqrt{(\frac{a}{\gamma})^2 - 1}}$$

$$\Rightarrow x_2\cos\delta - x_1\sin\delta = \frac{x_{n-k+2}}{\sqrt{(\frac{a}{\gamma})^2 - 1}}$$

This is the equation of a plane passing through the origin so its restriction to the submanifold $(\mathbb{R} \times \mathbb{R} \times \{0\}^{n-k-1} \times \mathbb{R} \times \{0\}^{k-1}) \cap M_{(n)}$ appears as a great circle of the sphere.

Notice that we get equation (2.3) from equation (2.2) simply by multiplying the left and right sides of equation (2.2) by $\frac{du_1}{du_k}$. This means that the best we can get from the system of equations we have created is a sectional solution of the curve. To obtain the general solution, we must set the arclength function in a single parameter. For instance,

$$I = \int L(u_1, u_2(u_1), \cdots, u_{n-1}(u_1), u_n(u_1), u'_2, \cdots u'_n)du_1,$$

where $u'_i = \frac{du_i}{du_1}$. By the canonical Euler - Lagrange equations, we have

$$\frac{\partial L}{\partial u_i} - \frac{d}{du_1}\left(\frac{\partial L}{\partial u'_i}\right) = 0 \text{ for } 2 \leq i \leq n.$$

(In problems where we have not provided explicit analytical solutions, the outlines are laid for alternative numerical approximation techniques in such instances. These can be implemented with appropriate computerized software such as MAPLE, MATLAB and MATHEMATICA.)

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