Monotone Operators and Applications

A Thesis
Submitted to the African University of Science and Technology
Abuja-Nigeria
in partial fulfilment of the requirements for

MASTER DEGREE IN PURE AND APPLIED MATHEMATICS

By

Usman Bello Abdulmalik

Supervisor:

Dr Guy Degla.

African University of Science and Technology
Abuja
Knowledge is Freedom

African University of Science and Technology
www.aust.edu.ng
P.M.B 681, Garki, Abuja F.C.T
Nigeria.

December 2011

Dedication

I dedicate this project to my beloved father and brother, Alhaji Abdulmalik Bello, Sani A. Bello. (May their soul rest in perfect peace).

Also my dedication goes to my mother Hajiya Bilkisu Ahamad, and to all brothers, sisters, and friends for their caring and support during my stay at African University of Science and Technology (AUST).

Preface

This project is mainly focused on the theory of *Monotone (increasing) Operators and its applications.* Monotone operators play an important role in many branches of Mathematics such as Convex Analysis, Optimization Theory, Evolution Equations Theory, Variational Methods and Variational Inequalities.
Basic examples of monotone operators are positive semi-definite matrices A of order $n \in \mathbb{N}$ (since they define linear operators on $\mathbb{R}^n$ and satisfy $\langle Ax, x\rangle \geq 0$ for all $x \in \mathbb{R}^n$), projection operators p_C onto closed convex nonempty subsets C of a Hilbert space (since $\langle x-y, p_C(x)-p_C(y)\rangle \geq 0$ for all $x, y \in H$), the derivative Df of a differentiable convex function f defined in a Banach space (since $\langle x-y, Df(x)-Df(y)\rangle \geq 0$ for all $x, y \in \mathrm{dom}(f)$), and the elliptic differential operator $-\Delta$ on $H^2(\mathbb{R}^n)$.
Monotone operators which have no proper monotone extension are called *maximal monotone operators* and are of particular interest because they are crucial in the solvability of evolution equations in Hilbert spaces as they generate semigroup of bounded linear operators.

In this project, we first study some fundamental geometric properties of Banach spaces, the topological properties of the duality mapping $J : X \to 2^{X^*}$ of a Banach space X defined by

$$J(x) \;:=\; \big\{ f \in X^* : \; \langle x, f\rangle = \|x\|\|f\| \text{ and } \|f\| = \|x\| \big\}$$

and after recalling some results from Convex Analysis, we observe that J is the subdifferential of the convex functional $x \mapsto \frac{||x||}{2}^2$.
Secondly we study the general properties of maximal monotone operators. In the case of a reflexive Banach space X, we have remarkable results such as:

- The Rockerfella's characterization of the maximality of a monotone (single or multi- valued) operator defined on a reflexive Banach space, which says that an operator A is maximal monotone on X if and only if $R(A + \lambda J) = X^*$, for some $\lambda > 0$.

- Every monotone and hemicontinuous operator is maximal monotone.

- Every maximal monotone operator which is coercive on X (e.g., J) is surjective.

Lastly we present another version of the Rockerfella's characterization with a shorter proof following C. Simons and C. Zalinescu [7], and we mention some applications of the surjectivity result for monotone operators to nonlinear elliptic equations in the line of Lions [10].

Acknowledgements

All praise is due to Allah, peace and blessing be upon the holy prophet Muhammad, his family, his faithful companions and those who follow their footstep till the day of judgement.

I am heartily thankfull to my supervisor Dr. Guy Degla, whose encouragement, guidance, and support from the initial to the final level enabled me to develop an understanding of this work.

My regards extends to the acting president of African University of Science and Technology, Abuja, Professor Charles Ejike Chidumi, and to Dr. Ngalla Djitte, Dr. Bashir Ali, Malam Ma'aruf Shehu Minjibir, Ahamad A. Bello for their support and advice during my stay at AUST.

Futhermore I am pleased to express my regards and blessings to my parents, brothers, sisters and all those who supported me in any respect during the work of this project.

Contents

1 Preliminaries **7**
1.1 Geometry of Banach Spaces 7
1.1.1 Uniformly Convex Spaces 7
1.1.2 Strictly Convex Spaces 9
1.1.3 Duality Mappings. 10
1.1.4 Duality maps of L^p Spaces $(p > 1)$ 13
1.2 Convex Functions and Subdifferentials 15
1.2.1 Basic notions of Convex Analysis 15
1.2.2 Subdifferential of a Convex function 19
1.2.3 Jordan Von Neumann Theorem for the Existence of Saddle point 20

2 Monotone operators. Maximal monotone operators. **23**
2.1 Maximal monotone operators 23
2.1.1 Definitions, Examples and properties of Monotone Operators 23
2.1.2 Rockafellar's Characterization of Maximal Monotone Operators 27
2.1.3 Topological Conditions for Maximal Monotone Operators 35
2.2 The sum of two maximal monotone operators 37
2.2.1 Resolvent and Yosida Approximations of Maximal Monotone Operators 37
2.2.2 Basic Properties of Yosida Approximations 38

3 On the Characterization of Maximal Monotone Operators **46**
3.1 Rockafellar's characterization of maximal monotone operators. 46

4 Applications **51**
4.1 Laplacian 51
4.2 Uniformly Monotone Operators 52

Chapter 1

Preliminaries

The aim of this chapter is to provide some basic results pertaining to geometric properties of normed linear spaces and convex functions. Some of these results, which can be easily found in textbooks are given without proofs or with a sketch of proof only.

1.1 Geometry of Banach Spaces

Throughout this chapter X denotes a real norm space and X^* denotes its corresponding dual. We shall denote by the pairing $\langle x, x^* \rangle$ the value of the function $x^* \in X^*$ at $x \in X$. The norm in X is denoted by $\| \cdot \|$, while the norm in X^* is denoted by $\| \cdot \|_*$. If there is no danger of confusion we omit the asterisk from the notation $\| \cdot \|_*$ and denote both norm in X and X^* by the symbol $\| \cdot \|$.

As usual We shall use the symbol $\to$ and $\rightharpoonup$ to indicate strong and weak convergence in X and X^* respectively. We shall also use w^*-lim to indicate the weak-star convergence in X^*. The space X^* endowed with the weak-star topology is denoted by X^*_w

1.1.1 Uniformly Convex Spaces

Definition 1.1. Let X be a normed linear space. Then X is said to be *uniformly convex* if for any $\varepsilon \in (0, 2]$ there exist a $\delta = \delta(\varepsilon) > 0$ such that for each $x, y \in X$ with $\|x\| \leq 1$, $\|y\| \leq 1$, and $\|x - y\| \geq \varepsilon$, we have $\|\frac{1}{2}(x + y)\| \leq 1 - \delta$.

Theorem 1.2. *Let X be a uniformly convex space. Then for any $d > 0, \varepsilon > 0$ and $x, y \in X$ with $\|x\| \leq d$, $\|y\| \leq d$, and $\|x - y\| \geq \varepsilon$, there exist a $\delta = \delta(\frac{\varepsilon}{d}) > 0$ such that $\|\frac{1}{2}(x + y)\| \leq (1 - \delta)d$.*

Proof. Let $x, y \in X$, set $k_1 = \frac{x}{d}, k_2 = \frac{y}{d}$ and $\overline{\varepsilon} = \frac{\varepsilon}{d}$. Then obviously we see that $\overline{\varepsilon} > 0$. Moreover , $\|k_1\| \leq 1, \|k_2\| \leq 1$ and $\|k_1 - k_2\| \geq \frac{\varepsilon}{d} = \overline{\varepsilon}$. Now, by the uniform convexity of X, we have for some $\delta(\frac{\varepsilon}{d}) > 0$,

$$\left\|\frac{1}{2}(k_1 + k_2)\right\| \leq 1 - \delta(\overline{\varepsilon}),$$

that is,

$$\left\|\frac{1}{2d}(x + y)\right\| \leq 1 - \overline{\varepsilon}(\delta),$$

which implies,

$$\left\|\frac{1}{2}(x + y)\right\| \leq [1 - \delta(\frac{\varepsilon}{d})]d.$$

Hence we have the result. □

Proposition 1.3. *Let X be a uniformly convex space, let $\alpha \in (0, 1)$ and $\varepsilon > 0$, then for any $d > 0$, $x, y \in X$ such that $\|x\| \leq d$, $\|y\| \leq d$, and $\|x - y\| \geq \varepsilon$ there exist $\delta(\varepsilon) > 0$ independent of x and y such that*

$$\|\alpha x + (1 - \alpha)y\| \leq [1 - 2\delta(\varepsilon)\ min\{\alpha, 1 - \alpha\}]d.$$

Proof. Without loss of generality we shall assume that $\alpha \in (0, \frac{1}{2}]$, we also observe that

$$\|\alpha x+(1-\alpha)y\| = \|\alpha(x + y)+(1-2\alpha)y\| \leq 2\alpha\|\frac{1}{2}(x+y)\|+(1-2\alpha)\|y\|$$

Thus from the uniform convexity of X we have for some $\delta(\varepsilon) > 0$

$$\begin{aligned}\|\alpha x + (1 - \alpha)y\| &\leq 2\alpha\left\|\frac{1}{2}(x + y)\right\| + (1 - 2\alpha)\|y\| \\ &\leq 2\alpha(1 - \delta(\varepsilon))d + (1 - 2\alpha)d \\ &= (1 - 2\alpha\delta(\varepsilon))d \\ &\leq [1 - 2\delta(\overline{\varepsilon})min\{\alpha, 1 - \alpha\}]d.\end{aligned}$$

Which completes the proof. □

1.1.2 Strictly Convex Spaces

Definition 1.4. A normed linear space X is said to be *strictly convex* if for all $x, y \in X$ $x \neq y$, $\|x\| = \|y\| = 1$, we have

$$\|\alpha x + (1-\alpha)y\| < 1 \text{ for all } \alpha \in (0,1).$$

Theorem 1.5. *Every uniformly convex space is strictly convex.*

Proof. Suppose X is uniformly convex, since $x \neq y$, set $\varepsilon = \|x-y\| > 0$ and $d = 1$. Then in view of proposition (1.3) we see that for each $\alpha \in (0,1)$, $\quad \|\alpha x + (1-\alpha)y\| < 1$, which gives the desired result.

We now give some examples to illustrate uniformly and strictly convex spaces.

Example 1. *Every inner product space H is uniformly convex. In particular $\mathbb{R}^n$ with the euclidean norm is uniformly convex.*

To see this we shall apply parallelogram law which is valid in any inner product space. That is for all $x, y, \in H$, we have

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

Now let $\varepsilon \in (0,2]$ be given, let $x, y \in H$ such that $\|x\| \leq 1, \|y\| \leq 1$, and $\|x-y\| \geq \varepsilon$, then from the above identity we have

$$\left\|\frac{1}{2}(x+y)\right\|^2 \leq \frac{1}{4}\left[2(2) - \|x-y\|^2\right] = 1 - \left\|\frac{1}{2}(x-y)\right\|^2 \leq 1 - \frac{1}{4}\varepsilon^2$$

So that

$$\left\|\frac{1}{2}(x+y)\right\| \leq \sqrt{1 - \frac{1}{4}\varepsilon^2}$$

To complete the proof we choose $\delta = \sqrt{1 - \frac{1}{4}\varepsilon^2} > 0$.

Example 2. $\mathbb{R}^n$ *with* $\|\cdot\|_1$ *is not strictly convex.* To see this we choose the canonical bases e_1, e_2 in $\mathbb{R}^n$ and take $\lambda = \frac{1}{2}$. Clearly $\|e_1\| = \|e_2\| = 1$, $e_1 \neq e_2$ and

$$\left\|\frac{1}{2}e_1 + \frac{1}{2}e_2\right\| = \frac{1}{2}\|e_1 + e_2\| = 1.$$

Thus we have $\mathbb{R}^n$ with $\|\cdot\|_1$ is not strictly convex.

Example 3. *The space $C[a,b]$ of all real valued continuous functions on the compact interval $[a,b]$ endowed with the "sup norm" is* *not strictly convex.* To see this we choose two functions such that

$$f(t) := 1 \text{ for all } t \in C[a,b], \quad g(t) := \frac{b-t}{b-a} \text{ for all } t \in C[a,b].$$

Take $\varepsilon = \frac{1}{2}$. Clearly, $f, g \in C[a,b]$, $\|f\| = \|g\| = 1$ and $\|f-g\| = 1 > \varepsilon$. But $\|\frac{1}{2}(x+y)\| = 1$. Thus, $C[a,b]$ is not strictly convex.

Theorem 1.6. *Let X be a reflexive Banach space with norm $\|\cdot\|$. Then there exist an equivalent norm $\|\cdot\|_0$ such that X is strictly convex in this norm and X^* is strictly convex in the dual norm $\|\cdot\|_0^*$.*

1.1.3 Duality Mappings.

Definition 1.7. Define a map $J : X \longrightarrow 2^{X^*}$ by

$$Jx \ := \ \Big\{x^* \in X^* : \langle x, x^*\rangle = \|x\|\|x^*\|;\ \|x\| = \|x^*\|\Big\}.$$

By Hahn Banach theorem for each $x \in X,\ x \neq 0$, there exist $y^* \in X^*$ such that $\|y^*\| = 1$, and $\langle x, y^*\rangle = \|x\|$. So if we set $x^* = \|x\|y^*$, then we see that for each $x \in X\ \exists\ x^* \in X^*$ such that $\|x^*\| = \|x\|$ and $\langle x, x^*\rangle = \|x\|^2$. So we see that for each $x \in X, Jx \neq \emptyset$. The mapping $J : X \longrightarrow 2^{X^*}$ is called the *duality mapping* of the space X. In general J is multivalued.

Remark. More generally, given an increasing continuous function $\varphi : [0, +\infty) \to [0, +\infty)$ such that $\varphi(0) = 0$ and $\lim_{+\infty} \varphi = +\infty$, one defines the duality map J_φ corresponding to the (normalization) function φ, by

$$J_\varphi x \ := \ \Big\{x^* \in X^* : \langle x, x^*\rangle = \|x\|\|x^*\|;\ \|x^*\| = \varphi(\|x\|)\Big\}.$$

Proposition 1.8. *Let H be a real Hilbert space and identify H^* with H, then*

$$Jx = \{x\} \quad \textit{for all } x \in H;$$

i.e The duality map J is the identity map.

Proof. Let $a \in H$. Define

$$\varphi_a(x) \ = \langle a,\ x\rangle \quad \forall x \in H.$$

Then $\varphi_a \in H^*$, $\|\varphi_a\| = \|a\|$ and $\varphi_a(a) = \|a\|^2$. Therefore $\varphi_a \in J(a)$and since φ_a is identified with a, via Riesz representation theorem, we can write $a \in J(a)$. Conversely, if $y \in Ja$ then $\langle a, y\rangle = \|a\|\|y\|$ and $\|a\| = \|y\|$ so that

$$\|a-y\|^2 = \langle a-y, a-y\rangle = \|a\|^2 + \|y\|^2 - 2\langle a, y\rangle = 2\|a\|^2 - 2\|y\|^2 = 0.$$

So we have $y = a$. Therefore $Ja = \{a\}$.

Proposition 1.9. *Let X be a real Banach and J be the duality mapping on X, then*

$$J(\lambda x) = \lambda J(x) \ \forall\ \lambda \in \mathbb{R} \quad \forall\ x \in X.$$

Proof. Let $y^* \in J(x)$ and $\lambda \in \mathbb{R}$. For $\lambda = 0$ the result follows trivially. suppose $\lambda \neq 0$, then we have

$$\langle \lambda x, \lambda y^*\rangle = \lambda^2\langle x, y^*\rangle = \|\lambda x\|\|\lambda y^*\|, \text{we also have } \|\lambda x\| = \|\lambda y^*\|.$$

Thus we have $\lambda y^* \in J(\lambda x)$, which implies $\lambda J(x) \subset J(\lambda x)$. From the preceding inclusion we also obtained that $\frac{1}{\lambda}J(\lambda x) \subset J(x)$ which implies $J(\lambda x) \subset \lambda J(x)$. Therefore $J(\lambda x) = \lambda J(x)\ \forall \lambda \in \mathbb{R}$, $\forall x \in X$.

Definition 1.10. Let $f : X \longrightarrow Y$ be a map. Then f is said to be *demi-continuous* if it is norm to weak-star continuous, i.e f is continuous from X (endowed with the strong topology) to Y (endowed with the weak-star topology).

Proposition 1.11. *Let X be a real norm space and J be the duality mapping on X. Then the following are true.*

a) *For each $x \in X$, Jx is a closed, convex subset of $B^*[0, \|x\|]$ in X^*. Where $B^*[0, \|x\|] = \{y^* \in X^* : \|y^*\| \leq \|x\|.\}$*

b) *If X^* is strictly convex, then for each $x \in X$, Jx is single valued. Moreover the mapping J is demi-continuous, i.e J is continuous as a mapping from X with the strong topology to X^* with the weak-star topology.*

c) *If X^* is uniformly convex, then for each $x \in X$, Jx is single valued and the mapping $x \longmapsto Jx$ is uniformly continuous on bounded subsets of X.*

Proof. (a) Obviously we have $Jx \subset B^*[0, \|x\|]$. Let $\{y_n^*\}_{n\geq 1} \subset Jx$ such that $y_n^* \to y$, for each $n \geq 1$ we have $\langle x, y_n^*\rangle = \|x\|\|y_n^*\|$ and $\|x\| = \|y_n^*\|$. Letting $n \to +\infty$ we see that $\langle x, y\rangle = \|x\|\|y\|$ and

$\|x\| = \|y\|$. Hence we have $y \in Jx$, which implies that Jx is closed.

For convexity, let $x^*, y^* \in Jx$ and $\lambda \in (0,1)$, then

$$\begin{aligned}\langle x, \lambda x^* + (1-\lambda)y^*\rangle &= \lambda\langle x, x^*\rangle + (1-\lambda)\langle x, y^*\rangle \\ &= \lambda\|x\|\|x^*\| + (1-\lambda)\|y^*\| = \|x\|^2\end{aligned}$$

We also have from the triangular inequality that $\|\lambda x^* + (1-\lambda)y^*\| \leq \|x\|$, also,

$$\begin{aligned}\|x\|^2 &= \langle x, \lambda x^* + (1-\lambda)y^*\rangle \\ &\leq \|x\|\|\lambda x^* + (1-\lambda)y^*\|,\end{aligned}$$

which implies that $\|x\| \leq \|\lambda x^* + (1-\lambda)y^*\|$. Hence we have

$$\|x\| = \|\lambda x^* + (1-\lambda)y^*\|, \text{ which shows that } \lambda x^* + (1-\lambda)y^* \in Jx.$$

(b) Assume X^* is strictly convex, and suppose that there exit $x^*, y^* \in Jx$ such that $x^* \neq y^*$, then $\|x^*\| = \|y^*\| = \|x\|$ and by the strict convexity of X we have that for any $\lambda \in (0,1)$, $\|\lambda x^* + (1-\lambda)y^*\| < \|x\|$. In particular taking $\lambda = \frac{1}{2}$, we have $\|\frac{1}{2}(x^* + y^*)\| < \|x\|$, which contradicts the fact that $\|\frac{1}{2}(x^* + y^*)\| = \|x\|$. (Since Jx is convex)

Let $\{x_n\}_{n\geq 1} \subset X$ such that $x_n \to x$. using the fact that $\|Jx_n\| = \|x_n\|$, i.e $\{Jx_n\}_{n\geq 1}$ is bounded and the fact that the unit ball is w^*-compact in X^* (Banach Alaoglo Theorem) we see that there exist a limit point y^* of $\{Jx_n\}_{n\geq 1}$. Now let $\{Jx_{n_k}\}_{k\geq 1} \subset X^*$ such that $w^* - limJx_{n_k} = y^*$, then we have $\lim_{k\to\infty} \langle x_{n_k}, Jx_{n_k}\rangle = \langle x, y^*\rangle$. We also have that

$$\lim_{k\to\infty} \langle x_{n_k}, Jx_{n_k}\rangle = \lim_{k\to\infty} \|x_{n_k}\|^2 = \|x\|^2.$$

So we get $\langle x, y^*\rangle = \|x\|^2$, which implies $\|x\| \leq \|y^*\|$. To get the reverse inequality we use the fact that $w^* - limJx_{n_k} = y^*$ implies $\|y^*\| \leq \liminf \|Jx_{n_k}\| = \liminf \|x_{n_k}\| = \|x\|$. Thus we have $\|x\| = \|y^*\|$ and $\langle x, y^*\rangle = \|x\|^2$. i.e $y^* = Jx$. Therefore J is demicontinuous.

(c) Since a uniformly convex space is also strictly, then by part (b) above we see that J is single valued.

Assume J is not uniformly continuous on bounded subsets of X, then there exist a constant $M > 0$, $\alpha_0 > 0$, and subsequences $\{u_n\}, \{v_n\} \subset X$ such that

$$\begin{aligned}&\|u_n\| \leq M, \ \|v_n\| \leq M, \ n \geq 1, \\ &\|u_n - v_n\| \to 0 \ as \ n \to \infty, \\ &\|Ju_n - Jv_n\| \geq \alpha_0, \ n \geq 1. \end{aligned} \tag{1.1}$$

Let $\beta > 0$ such that $\|u_n\| \geq \beta,\ \|v_n\| \geq \beta,$ for $n \geq 1$. Such β exist, for if there exist a subsequence $\{u_{n_k}\} \subset X$ such that $u_{n_k} \to 0$ as $n \to +\infty$, then we see that $v_{n_k} \to 0$. From the definition of duality map we obtained that $Ju_{n_k} \to 0$ and $Jv_{n_k} \to 0$, and this contradicts (1.1).

Now set

$$x_n = \frac{u_n}{\|u_n\|},\ y_n = \frac{v_n}{\|v_n\|}\ u_n, v_n\ \neq 0. \text{ Then we have,}$$

$$\begin{aligned}\|x_n - y_n\| &= \frac{1}{\|u_n\|\|v_n\|}\|u_n\|v_n\| - \|u_n\|v_n\| \\ &\leq \frac{1}{\beta^2}(\|v_n\|\|u_n - v_n\| + \|\|v_n\| - \|u_n\|\|\|v_n\|) \\ &\leq \frac{2M}{\beta^2}\|u_n - v_n\| \to 0 \ as\ n \to +\infty\end{aligned}$$

We also have $2 \geq \|Jx_n + Jy_n\| \geq \langle x_n, Jx_n + Jy_n\rangle$ which together with

$$\begin{aligned}\langle x_n, Jx_n + Jy_n\rangle &= \|x_n\|^2 + \|y_n\|^2 + \langle x_n - y_n, Jy_n\rangle \\ &= 2 + \langle x_n - y_n, Jy_n\rangle \geq 2 - \|x_n - y_n\|\end{aligned}$$

implies

$$\lim_{n\to\infty} \|Jx_n + Jy_n\| = 2 \text{ i.e } \lim_{n\to\infty} \|\frac{1}{2}(Jx_n + Jy_n)\| = 1. \tag{1.2}$$

Now suppose there exist $\varepsilon_0 > 0$ and a subsequence $\{x_{n_k}\},\ \{y_{n_k}\} \subset X$ such that $\|Jx_{n_k} - Jy_{n_k}\| \geq \varepsilon_0$, for $n \geq 1$. Observing that $\|Jx_{n_k}\| = \|Jy_{n_k}\| = 1$ and using the uniform convexity of X^* we see that there exist $\delta(\varepsilon_0) > 0$ such that $\|\frac{1}{2}(Jx_{n_k}+Jy_{n_k})\| \leq 1-\delta(\varepsilon_0)$ which contradicts (1.2). Therefore we have $\lim\limits_{n\to\infty} \|Jx_n - Jy_n\| = 0$, which implies

$$\begin{aligned}\|Ju_n - Jv_n\| &= \|J(\|u_n\|x_n) - J(\|v_n\|y_n)\| \\ &= \|\|u_n\|Jx_n - \|v_n\|Jy_n\| \\ &\leq \|u_n\|\|Jx_n - Jy_n\| + \|v_n\|\|\|u_n\| - \|v_n\|\| \\ &\leq M\|Jx_n - Jy_n\| + \|u_n - v_n\| \to 0 \ as\ n \to +\infty.\end{aligned}$$

This contradicts (1.1). Hence we have the result. □

1.1.4 Duality maps of L^p Spaces ($p > 1$)

Proposition 1.12. *The duality map on* $L^p([0,1])$, $p > 1$ *is given by* $J(0) = \{0\}$ *and for* $f \neq 0$

$$J(f) \ = \ \{\phi_{f^*}\}$$

where $\phi_{f^*} \in (L^p)^* = L^q$ is defined by

$$\phi_{f^*}(g) \;=\; \int_0^1 f^*(t)g(t)dt \quad \forall g \in L^p$$

and

$$f^* = \frac{|f|^{p-1} signf}{\|f\|_p^{p-2}}$$

Observe that when $p \geq 2$, this f^* has the following expression:

$$f^* \;=\; \frac{f|f|^{p-2}}{||f||_p^{p-2}} \, .$$

Proof. Now set $A_f = \{\phi_{f^*}\}$. By definition of the duality map

$$J(f) = \{\phi \in (L^p)^* : \phi(f) = \|f\|\|\phi\|, \; \|f\| = \|\phi\|\}.$$

Let $\phi \in J(f)$, since $\phi \in (L^p)^*$, then by Riez representation theorem there exist a unique $f^* \in L^q$, $\frac{1}{p} + \frac{1}{q} = 1$ $p, q > 1$ such that

$$\phi(f) = \langle f, f^* \rangle = \int_0^1 f^*(t)f(t)\,\mathrm{d}t, \; \|\phi\| = \|f^*\|.$$

Setting $\phi = \phi_{f^*}$, we have

$$\phi_{f^*}(f) = \langle f, f^* \rangle = \int_0^1 f^*(t)f(t)\,\mathrm{d}t, \; \|\phi_{f^*}\| = \|f^*\|.$$

So we have

$$\|f\|\|f^*\| = \phi_f^*(f) = \int_0^1 f^*(t)f(t)\,\mathrm{d}t, \; \|\phi_f^*\| = \|f\| = \|f^*\|.$$

which implies

$$\int_0^1 f^*(t)f(t)\,\mathrm{d}t = \|f\|^2, \; \|f^*\|_q = \|f\|_p. \tag{1.5}$$

We now show that $f^*(t) := \frac{|f(t)|^{p-1} signf(t)}{\|f\|_p^{p-2}}$ satisfies (1.5). But we have

$$\begin{aligned}
\left(\int_0^1 |f^*(t)|^q \,\mathrm{d}t\right)^{\frac{1}{q}} &= \left(\int_0^1 \frac{|f(t)|^{q(p-1)}}{\|f\|^{(p-2)q}} \,\mathrm{d}t\right)^{\frac{1}{q}} \\
&= \frac{1}{\|f\|^{(p-2)}} \left(\int_0^1 |f(t)|^q \,\mathrm{d}t\right)^{\frac{1}{q}} \\
&= \frac{\|f\|^{\frac{p}{q}}}{\|f\|^{p-2}} = \frac{\|f\|^{p-1}}{\|f\|^{p-2}} = \|f\|_p.
\end{aligned}$$

Therefore $\|f^*\|_q = \|f\|_p$. Also

$$\begin{aligned}\int_0^1 f^*(t)f(t)\,\mathrm{d}t &= \int_0^1 \frac{|f(t)|^{p-1}signf(t)}{\|f\|_p^{p-2}} f(t)\,\mathrm{d}t \\ &= \frac{1}{\|f\|^{p-2}} \int_0^1 |f(t)|^p\,\mathrm{d}t \\ &= \frac{\|f\|^p}{\|f\|^{p-2}} = \|f\|_p\|f\|_p = \|f\|_p\|f^*\|_q = \|f\|\|\phi_{f^*}\|.\end{aligned}$$

Thus $J(f) \subset A_f$. On the other hand for arbitrary $h \in L^p([0,1])$,

$$\phi_{f^*}(h) = \int_0^1 f^*(t)h(t)\,\mathrm{d}t,\ \|\phi_{f^*}\| = \|f^*\|.$$

In particular

$$\begin{aligned}\phi_{f^*}(f) &= \int_0^1 g(t)f(t)\,\mathrm{d}t \\ &= \int_0^1 f(t)\frac{|f(t)|^{p-1}signf(t)}{\|f\|_p^{p-2}}\,\mathrm{d}t \\ &= \frac{1}{\|f\|^{p-2}} \int_0^1 |f(t)|^p\,\mathrm{d}t = \|f\|_p^2.\end{aligned}$$

So we have $\phi_{f^*}(f) = \|f\|_p\|f\|_p$ and $\|f\|_p = \|f^*\|_q$ so that $\|\phi f^*\| = \|f\|_p$. Thus $A_f \subset J(f)$. Therefore $A_f = J(f)$. □

1.2 Convex Functions and Subdifferentials

In this section we present the basic properties of convex functions and subdifferentials as we shall use them in the next chapter.

1.2.1 Basic notions of Convex Analysis

Definition 1.13. Let C be a non empty subset of a real norm linear space X. The set C is said to be *convex* if for each $x, y \in C$ and for each $t \in (0,1)$ we have $tx + (1-t)y \in C$.

Definition 1.14. Let C be a non empty convex subset of X. Then *the convex hull* of C denoted by *coC* is the intersection of all convex sets containing C. (Equivalently, convex hull of C is the set of all convex combinations of finite subsets of points of C).

Definition 1.15. Let C be a non empty convex subset of X. Let $f : C \longrightarrow \mathbb{R} \cup \{+\infty\}$. Then f is said to be *convex* if for each $t \in (0, 1)$ and for all $x, y \in C$ we have

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y).$$

Moreover f is said to be proper if f is not identically $+\infty$ (i.e $\exists\, x_0 \in C$ such that $f(x_0) \in \mathbb{R}$)

Definition 1.16. Let $f : C \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a map. The *effective domain* of f is defined by

$$D(f) = \{x \in X : f(x) < +\infty\}.$$

The set

$$epi(f) = \{(x, \alpha) \in X \times \mathbb{R} : f(x) \leq \alpha\}$$

is called the epigraph of f, while

$$S_\alpha = \{x \in X : f(x) \leq \alpha\}$$

is called the section of f.

Proposition 1.17. *A mapping $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is convex if and only if its epigraph is convex.*

Proposition 1.18. (Slope Inequality) *Let I be an interval of $\mathbb{R}$ and $f : I \longrightarrow \mathbb{R}$ be a convex function. Assume $r_1 < r_2 < r_3$ with $r_i \in I$ for $i = 1, 2, 3$. and $f(r_1), f(r_2)$ are finite. Then*

$$\frac{f(r_2) - f(r_1)}{r_2 - r_1} \leq \frac{f(r_3) - f(r_1)}{r_3 - r_1} \leq \frac{f(r_3) - f(r_2)}{r_3 - r_2}.$$

Proposition 1.19. *Suppose $f : I \longrightarrow \mathbb{R}$ is convex and derivable on I. Then f' is increasing.*

Proof. Let $r < t$ we show that $f'(r) \leq f'(t)$. Now

$$f'(r) = \lim_{s \to r^+} \frac{f(s) - f(r)}{s - r} \leq \frac{f(t) - f(r)}{t - r} \leq \lim_{s \to r^-} \frac{f(s) - f(t)}{s - t} = f'(t)$$

Definition 1.20. Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a map. Let $x_0 \in D(f)$, then f is *lower semicontinuous* at x_0 if for each $\varepsilon > 0$ there exist $\delta > 0$ such that $f(x_0) - \varepsilon < f(x)$ for all $x \in B(x_0, \delta)$.

Proposition 1.21. *Let $f : X \longrightarrow \mathbb{R}\cup\{+\infty\}$ be a map. Let $x_0 \in D(f)$, then f is lower semicontinuous at x_0 if and only if*

$$\liminf f(x_n) \geq f(x_0)$$

for all $\{x_n\} \subset X$ such that $x_n \to x_0$.

Proposition 1.22. *Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a map. Then the following are equivalent.*
(a) f is lower semicontinuous,
(b) $epi(f)$ is closed,
(c) S_α is closed for each $\alpha \in \mathbb{R}$.

Definition 1.23. Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a map. Then f is said to be *coercive* if

$$\lim_{\|x\|\to\infty} f(x) = +\infty.$$

Proposition 1.24. *Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a map. Then f is convex and l.s.c if and only if f is convex and weakly l.s.c.*

Proof.

$$\begin{aligned} \text{f is convex and l.s.c} &\Leftrightarrow epi(f) \text{ is convex and closed} \\ &\Leftrightarrow epi(f) \text{ is convex and weakly closed} \\ &\Leftrightarrow f \text{ is convex and weakly l.s.c.} \end{aligned}$$

Theorem 1.25. *Assume X is reflexive. Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be proper, convex, coercive and l.s.c function on X. Then f has a minimum on X. That is there exist $x_0 \in X$ such that*

$$f(x_0) = \inf_{x\in X} f(x).$$

Proof. Let $\eta = \inf\limits_{x\in X} f(x)$. Since f is proper we see that $\eta < +\infty$. Let $\{x_n\} \subset X$ such that $f(x_n) \to \eta < +\infty$, then from the coercivity condition of f we see that $\{x_n\}$ is bounded. Since X is reflexive, then there exist $x_0 \in X$ and a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \rightharpoonup x_0$. In view of proposition (1.24) f is weakly lower semi continuous. So we have

$$\eta \leq f(x_0) \leq \liminf f(x_{n_k}) = \lim_{k\to\infty} f(x_{n_k}) = \eta.$$

Therefore

$$f(x_0) = \eta = \inf_{x\in X} f(x).$$

Theorem 1.26. *Let f be proper, convex, and lower semi-continuous on X. Then f is bounded from below by an affine function. i.e There exist $x^* \in X^*$ and a constant $c \in \mathbb{R}$ such that*

$$f(x) \geq \langle x, x^* \rangle + c \text{ for all } x \in X.$$

Proof. Let $x_0 \in D(f)$ and $\beta \in \mathbb{R}$ such that $f(x_0) > \beta$. This is possible since f is proper, i.e $D(f) \neq \emptyset$. Clearly $(x_0, \beta) \notin X \times \mathbb{R}$, also in view of proposition (1.22) $epi(f)$ is closed and convex, so by Hahn Banach theorem there exists a closed hyperplane

$$H = \{(x, \lambda) \in X \times \mathbb{R} : \langle x, x_0^* \rangle + \gamma\lambda = \alpha\}$$

that separates $epi(f)$ and (x_0, β) i.e

$$\langle x, x_0^* \rangle + \gamma\lambda \leq \alpha \leq \langle x_0, x_0^* \rangle + \gamma\lambda$$

Considering the left hand side of the inequality only, it is easy to see that $\gamma < 0$, otherwise we arrived at contradiction. Therefore we have

$$\lambda \geq \frac{\alpha}{\gamma} + \langle x, -\frac{x_0}{\gamma} \rangle \text{ for all } (x, \lambda) \in epi(f).$$

Since for each x in X $(x, f(x)) \in epi(f)$, we see that

$$f(x) \geq \langle x, x^* \rangle + c \text{ for all } x \in X, \text{ where } x^* = -\frac{x_0}{\gamma} \text{ and } c = \frac{\alpha}{\gamma}.$$

Definition 1.27. Let X be Banach space. Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a function. Let $x \in D(f)$ and $v \in X$, then we say that f has a *directional derivative* at x in the direction of $v \neq 0$ if the limit

$$\lim_{t\to 0^+} \frac{f(x + tv) - f(x)}{t} \text{ exist.}$$

We denote by $f^{'}(x, v)$ the directional derivative of f at x in the direction of v, and we write

$$f^{'}(x, v) = \lim_{t\to 0^+} \frac{f(x + tv) - f(x)}{t}.$$

The function $f : X \longrightarrow \mathbb{R}$ is said to be Gâteaux differentiable at $x \in X$ if for all $v \in X$ $f^{'}(x,v)$ exists in $\mathbb{R}$ and the function $v \mapsto f^{'}(x,v)$ is linear and continuous. We denote by $D_G f(x)$ the Gâteaux differential of f at x and

$$\langle D_G f(x), v\rangle := f^{'}(x,v) \text{ for all } v \in X.$$

1.2.2 Subdifferential of a Convex function

Definition 1.28. Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper and convex function. Let $x \in D(f)$, then the *subdifferential* $\partial f(x)$ of f at x is the set

$$\partial f(x) = \{x^* \in X^* : \langle y - x, x^*\rangle \leq f(y) - f(x) \ \forall y \in X\}.$$

We remarked that if x is not in $D(f)$ then $\partial f(x) = \emptyset$.

Proposition 1.29. *Let X be proper and convex function which is Gâteaux differentiable at $x \in D(f)$ then*

$$\partial f(x) = \{D_G f(x)\}.$$

Proof. To see this we pick $y \in X$. Convexity of f implies that

$$\frac{f(x + t(y - x)) - f(x)}{t} \leq f(y) - f(x), \quad 0 < t < 1.$$

Since f is Gâteaux differentiable at x we obtained that

$$\langle y - x, D_G f(x)\rangle = \lim_{t \to 0^+} \frac{f(x + t(y - x)) - f(x)}{t} \leq f(y) - f(x).$$

Thus $D_G f(x) \in \partial f(x)$.
Conversely, let $w^* \in \partial f(x)$, then for any $y \in X$ and $t > 0$

$$\frac{f(x + ty) - f(x)}{t} \geq \langle y, w^*\rangle.$$

Using the Gâteaux differentiability of f at x we obtained that

$$\langle y, D_G f(x)\rangle \geq \langle y, w^*\rangle \quad \forall y \in X$$

which implies $D_G f(x) = w^* \in \partial f(x)$. Therefore $\partial f(x) = \{D_G f(x)\}$.

Example 1. *Define a function $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ by*

$$f(x) = \frac{1}{2}\|x\|^2 \quad \forall x \in X.$$

Then f is proper, convex and continuous. Moreover $\partial f(x) = J(x)$ for each $x \in X$ where J is the duality map on X.

Indeed choose first $x^* \in Jx$. Then for any $y \in x$ we have

$$\begin{aligned}\langle y - x, x^* \rangle = \langle y, x \rangle - \|x\|^2 &\leq \|y\|\|x\| - \|x\|^2 \\ &\leq \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x\|^2 \\ &= f(y) - f(x).\end{aligned}$$

Thus we have $x^* \in \partial f(x)$. Conversely, for $x^* \in \partial f(x)$ we have

$$\langle y - x, x^* \rangle \leq f(y) - f(x) \ \forall y \in X.$$

So considering $x + ty$, $t \in (0, 1)$ we get

$$\langle x^*, y \rangle \leq \frac{1}{2t}(\|x + ty\|^2 - \|x\|^2) \leq \|x\|\|y\| + \frac{t}{2}\|y\|.$$

As $t \to 0^+$ we have $\langle x^*, y \rangle \leq \|x\|\|y\|$, which implies $\|x^*\| \leq \|x\|$. Also using the fact that $x^* \in \partial f(x)$ and considering $x + tx \in X$ we have

$$2t\langle -x, x^* \rangle \leq \|x - tx\|^2 - \|x\|^2 = (t^2 - 2t)\|x\|^2, \quad t > 0.$$

So we have $(2 - t)\|x\|^2 \leq 2\langle x, x^* \rangle$. Now as $t \to 0^+$ we obtained

$$\|x\|^2 \leq \langle x, x^* \rangle \leq \|x\|\|x^*\| \text{ which implies } \|x\| \leq \|x^*\|.$$

Therefore we have $\|x\| = \|x^*\|$ and $\langle x, x^* \rangle = \|x\|^2$. Thus $x^* \in J(x)$.

Example 2. *Let K be a closed, convex subset of X. Define a map I_k on X by*

$$I_k(x) = \begin{cases} 0 & \text{for } x \in K, \\ +\infty & \text{for } x \notin K. \end{cases}$$

It is easy to see that I_k is convex and lower semi-continuous (since K is convex and closed). Futhermore for any $x \in K$ we get

$$\partial I_k(x) = \{x^* \in X^* : \langle y - x, x^* \rangle \leq 0, \ \forall y \in X\}.$$

1.2.3 Jordan Von Neumann Theorem for the Existence of Saddle point

We now state and prove Jordan von Neumann Theorem for the existence of saddle point for an upper semi-continuous function defined on a compact convex subset of a Banach space. But before that we state

Kakutani fixed point theorem without proof.

Theorem 1.30. (Kakutani) *Let K be a nonempty compact convex subset of a Banach space and let*

$$T:\ K \longrightarrow 2^K$$

be a mapping having a closed graph (i.e., T is upper semicontinuous) and such that for every $x \in K$, $T(x) \subset K$ is nonempty, closed and convex. Then there exists at least one $x \in K$ such that $x \in T(x)$.

Theorem 1.31. (J.Von Neumann) *Let X and Y be real Banach spaces and let $U \subset X$ and $V \subset Y$ be compact convex subsets of X and Y, respectively. Let $H : U \times V \longrightarrow \mathbb{R}$ be a continuous, convex-concave function (i.e $H(u, v)$ is convex as a function of u and concave as a function of v). Then there exists $(u_0, v_0) \in U \times V$ such that*

$$H(u_0, v) \ \leq\ H(u_0, v_0) \ \leq\ H(u, v_0) \qquad \forall\, u \in U \ \ \text{and}\ \ \forall\, v \in V.$$

Such a point $(u_0,\, v_0)$ is called the saddle point of the function H.

Proof. Define the mappings $T_1 : U \longrightarrow V$, $\ T_2 : V \longrightarrow U$ and

$$T:\ U \times V \ \longrightarrow\ 2^{U\times V}$$

respectively by

$$T_1(u) \ =\ \Big\{v \in V : H(u, v) \geq H(u, w) \ \ \forall\ w \in U\Big\},$$

$$T_2(v) \ =\ \Big\{u \in U : H(u, v) \leq H(w, v) \ \ \forall\ w \in V\Big\},$$

$$T(u, v) \ =\ T_2(v) \times T_1(u).$$

Since H is continuous we see that the graph of T;

$$\text{Graph}(T) \ =\ \Big\{\big((u, v),\, (x, y)\big) \in (U \times V)^2 \ :\ (x, y) \in T(u, v)\Big\}\,,$$

is closed. Also for each $(u, v) \in U \times V$, $T(u, v)$ is convex. To see this it is enough to show that $T_1(u)$ and $T_2(v)$ are convex. Let $u_1, u_2 \in T_2(v)$ and $\lambda \in (0, 1)$. Since H is convex as a function of u we see that

$$\begin{aligned} H(\lambda u_1 + (1-\lambda)u_2, v) &\leq \lambda H(u_1, v) + (1-\lambda)H(u_2, v) \\ &\leq \lambda H(w, v) + (1-\lambda)H(w, v) \\ &= H(w, v) \quad \ \forall\, w \in V. \end{aligned}$$

which implies $\lambda u_1 + (1-\lambda) u_2 \in T_2(v)$. So we have $T_2(v)$ is convex. Similarly we have $T_1(u)$ is convex, and hence $T(u,v)$ is convex. Therefore by theorem (1.30) there exists $(u_0, v_0) \in U \times V$ such that

$$(u_0, v_0) \in T(u_0, v_0) \;=\; T_2(v_0) \times T_1(u_0)$$

which implies

$$H(u_0, v) \;\leq\; H(u_0, v_0) \;\leq\; H(u, v_0) \quad \forall\, u \in U \text{ and } \forall\, v \in V.$$

The proof is complete. □

Chapter 2

Monotone operators. Maximal monotone operators.

In this chapter we introduce the concept of maximal monotone operators, alongside with the result concerning this theory and its relationship with convex analysis.

2.1 Maximal monotone operators

2.1.1 Definitions, Examples and properties of Monotone Operators

Throughout this chapter X will denote real Banach space with dual X^*. Notations for norms, convergence, and duality pairing will be the same as introduced in chapter 1, section 1.1. If X is Hilbert space we shall identify it with its own dual unless otherwise stated.

If X and Y are two linear spaces, we will denote by $X \times Y$ the cartesian product. The elements of $X \times Y$ will be written as (x, y) where $x \in X$ and $y \in X$. If A is multivalued operator from X to Y we may identify it with its graph in $X \times Y$. i.e

$$\{(x, y) \in X \times Y : y \in Ax\}.$$

Conversely, if $A \subset X \times Y$ then we define

$$Ax = \{y \in Y : (x, y) \in A\}, \quad D(A) = \{x \in X : Ax \neq \emptyset\},$$
$$R(A) = \bigcup_{x \in D(A)} Ax, \quad A^{-1} = \{(y, x) : (x, y) \in A\}.$$

where Ax, $D(A)$, $R(A)$, and A^{-1} are image of x, domain of A, range of A and inverse of A respectively. We shall identify operators from X

to Y with their graphs in $X \times Y$ and so we can equivalently speak of subsets of $X \times Y$ instead of operators from X to Y. If $A, B \subset X \times Y$ and λ is a real number, we set

$$\lambda A = \{(x, \lambda y) : (x, y) \in A\},$$
$$A + B = \{(x, y + z) : (x, y) \in A, (x, z) \in A\}.$$

Definition 2.1. Let $A \subset X \times X^*$ be multivalued operator. Then A is said to be *monotone* if

$$\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0 \ \forall \ (x_i, y_i) \ \in A. \ i = 1, 2.$$

A monotone set $A \subset X \times X^*$ is said to be *maximal monotone* if it is not properly contained in any other monotone subset of $X \times X^*$. We note that if A is single valued, then A is monotone if

$$\langle x_1 - x_2, Ax_1 - Ax_2 \rangle \geq 0 \ \forall \ x_1, x_2 \in D(A).$$

We now give some examples of monotone operators

Example 1. *Let X be a real Banach space. Then the duality map as defined in chapter 1, section 1.1. is monotone.*

Indeed for any $(x_i, y_i) \in J, \ i = 1, 2.$ we have

$$\begin{aligned} \langle x_1 - x_2, y_1 - y_2 \rangle &= \|x_1\|^2 + \|x_2\|^2 - \langle x_1, y_2 \rangle - \langle x_2, y_1 \rangle \\ &\geq \|x_1\|^2 + \|x_2\|^2 - \|x_1\|\|y_2\| - \|x_2\|\|y_1\| \\ &= \|x_1\|^2 + \|x_2\|^2 - 2\|x_1\|\|x_2\| \\ &= (\|x_1\| - \|x_2\|)^2 \geq 0. \end{aligned}$$

Example 2. *Every non decreasing function on $\mathbb{R}$ is monotone.*

To see this, let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a non decreasing function. Then for arbitrary $x, y \in \mathbb{R}$ with $x \leq y$ we have $f(x) \leq f(y)$. Thus we see that

$$\langle y - x, f(y) - f(x) \rangle \geq 0 \text{ for all } x, y \in \mathbb{R},$$

which shows the monotonicity of f.

Example 3. *Let A be $n \times n$ matrix with real entries. Consider the function $g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by $g(x) = Ax$. Then g is monotone if and only if A is positive semi definite.*

Example 4. *Let H be a real Hilbert space, I the identity map of H and $T : H \longrightarrow H$ be a non expansive map (i.e, $\|Tx - Ty\| \leq \|x - y\|$). Then the operator $I - T$ is monotone.*

It follows that every orthogonal projection of a Hilbert space is monotone.

Let $x, y \in H$, then

$$\begin{aligned}\langle x-y,(I-T)x-(I-T)y\rangle &= \langle x-y,(x-y)-(Tx-Ty)\rangle \\ &= \|x-y\|^2-\langle x-y,Tx-Ty\rangle \\ &\geq \|x-y\|^2-\|x-y\|\|Tx-Ty\| \\ &\geq \|x-y\|^2-\|x-y\|^2=0.\end{aligned}$$

Here we have used Cauchy inequality and the fact that T is non expansive. Thus we have $I-T$ is monotone on H.

Example 5. *Let U be an open convex subset of of a real Banach space X. Let $f : U \longrightarrow \mathbb{R}$ be convex and differentiable on U. Then $f' : U \longrightarrow X^*$ is monotone.*

Proof. Let $x, y \in U$. Define $I = \{s \in \mathbb{R} : x + s(y-x) \in U\}$.
Claim: I is an interval of $\mathbb{R}$ and $0, 1 \in I$.
To see this let $s_1, s_2 \in I$, $t \in (0,1)$, then we have

$$x + s_1(y-x) \in U \text{ and } x + s_2(y-x) \in U.$$

Now, since U is convex we have

$$ts_1 + (1-t)s_2 = t(x + s_1(y-x)) + (1-t)(x + s_2(y-x)) \in U.$$

So we see that I is an interval of $\mathbb{R}$. Now define $h : I \longrightarrow \mathbb{R}$ by

$$h(s) = f(x + s(y-x)).$$

Clearly h is convex and derivable (since f is convex and derivable). Thus we have from proposition (1.19) that h' is increasing on I. Thus $h'(1) \geq h'(0)$. We also observed that $h'(s) = \langle f'(x + s(y-x)), y-x\rangle$. Therefore we have

$$\begin{aligned}0 \leq h'(1)-h'(0) &= \langle f'(y), y-x\rangle - \langle f'(x), y-x\rangle \\ &= \langle f'(y)-f'(x), y-x\rangle.\end{aligned}$$

Which shows the monotonicity of f'.

Definition 2.2. Let A be a single valued operator. Then A is said to be *hemi-continuous* if it is weakly continuous in every direction, i.e if for all $x_1, x_2, x \in X$, the function $\mathbb{R} \longrightarrow \mathbb{R}$ defined by $\lambda \longmapsto \langle x, A(x_1 + \lambda x_2)\rangle$ is continuous on $\mathbb{R}$.

A is said to be coercive if

$$\lim_{n\to\infty} \frac{\langle x_n - x_0, y_n\rangle}{\|x_n\|} = +\infty. \tag{2.0}$$

for some $x_0 \in X$ and for all $(x_n, y_n) \in A$ such that $\lim_{n\to\infty} \|x_n\| = +\infty$.

Proposition 2.3. *Let $A \subset X \times X^*$ be maximal monotone. Then*
(a) A is weakly strongly closed. i.e if $(x_n, y_n) \in A$ such that $x_n \rightharpoonup x$ and $y_n \to y$, then $(x, y) \in A$.
(b) A^{-1} is maximal monotone.
(c) For each $x \in D(A)$, Ax is a closed, convex subset of X^*.

Proof. (a) Let $\{x_n\} \subset X$ such that $x_n \rightharpoonup x$ and $y_n \to y$. Then from the inequality

$$\langle x_n - u, y_n - v\rangle \geq 0 \;\forall\; (u, v) \in A,$$

we have

$$0 \leq \langle x_n - u, y_n - v\rangle = \langle x_n, y_n\rangle - \langle x_n, v\rangle - \langle x, y_n\rangle + \langle u, v\rangle.$$

Now using the weak and weak star convergence in X we see that

$$0 \leq \lim_{n\to\infty} \langle x_n - u, y_n - v\rangle = \langle x - u, y - v\rangle \;\forall\; (u, v) \in A.$$

Since A is maximal monotone we see that $(x, y) \in A$.

(b) This follows directly from the duality map and monotonicity of A.

(c) To show that Ax is a closed subset of X^* we set $x_n = x \;\forall\; n \geq 1$. Then we see that the results follows directly from (a) above. Now to show that Ax is a convex subset X^*. Let $x_0 \in X,\; y_1, y_2 \in Ax$ and $\lambda \in [0, 1]$. Then from the inequalities

$$\langle u - x_0, v - y_1\rangle \geq 0,\; \langle u - x_0, v - y_2\rangle \geq 0 \;\forall\; (u, v) \in A,$$

we have $\forall\; (u, v) \in A$,

$$\begin{aligned}\langle u - x_0, v - (\lambda y_1 + (1 - \lambda)y_2)\rangle &= \lambda\langle u - x_0, v - y_1\rangle \\ &\quad + (1 - \lambda)\langle u - x_0, v - y_2\rangle \geq 0.\end{aligned}$$

Now define

$$\hat{A}x = \begin{cases} Ax, & x \neq x_0, \\ Ax \cup \{\lambda y_0 + (1 - \lambda)y_1\}, & x = x_0. \end{cases}$$

Then we see that $\hat{A}$ is monotone, and since A is maximal monotone we conclude that $\hat{A} = A$, which implies

$$\lambda y_1 + (1 - \lambda) y_2 \in A x_0.$$

Hence Ax is a convex subset of X^* for each $x \in D(A)$. □

2.1.2 Rockafellar's Characterization of Maximal Monotone Operators

Before stating the Rockafellar's Theorem for the Characterization of Maximal Monotone Operators we first prove the following results.

Lemma 2.4. *Let X be a reflexive Banach space.*
(a) Assume that $M \subset X$ is bounded and $A \subset M \times X^*$ is monotone. Then for each $x^* \in X^*$ there exist $x \in \overline{coM}$ such that

$$\langle u - x, v - x^* \rangle \geq 0 \quad \forall\ (u, v) \in A.$$

(b) Let $A \subset X \times X^*$ be monotone and assume that the range of A denoted by $R(A)$ is bounded. Then for each $x \in X$, there exist $x^* \in \overline{coR(A)}$ such that

$$\langle u - x, v - x^* \rangle \geq 0 \quad \forall\ (u, v) \in A.$$

Proof. Let $x^* \in X^*$ be fixed. For each $(u, v) \in A$ define

$$X(u, v) = \{x \in \overline{coM} : \langle u - x, v - x^* \rangle \geq 0\}.$$

Since $u \in D(A) \subset M \subset \overline{coM}$, we easily see that $X(u, v) \neq \emptyset$. Also for each $(u, v) \in A$, $X(u, v)$ is a closed and convex subset of $\overline{coM}$. Moreover $\overline{coM}$ is a closed, convex and bounded subset of reflexive space, so its is weakly compact. Thus to show that

$$\bigcap_{(u,v) \in A} X(u, v) \neq \emptyset$$

it is enough to show that

$$\bigcap_{i=1}^{n} X(u_i, v_i) \neq \emptyset$$

for any finite number $(u_i, v_i) \in A,\ i = 1, 2, ...n$. Now consider the compact n-simplex convex subset of $\mathbb{R}^n$ C_n defined by

$$C_n = \{\alpha \in \mathbb{R}^n : \sum_{i=1}^{n} \alpha_i = 1, \alpha_i \geq 0\ \forall\ i = 1, 2, ...n\}.$$

Define a continuous function $\psi : C_n \times C_n \longrightarrow \mathbb{R}$ by

$$\psi(\alpha,\beta) = \sum_{i=1}^{n} \beta_i \langle x(\alpha) - u_i, v_i - x^* \rangle, \ \ \alpha, \beta \in C_n,$$

Where $x(\alpha) = \sum_{j=1}^{n} \alpha_j u_j$. Clearly for fixed β, ψ is convex and for fixed α, ψ is concave, so by theorem (1.31) there exist $\alpha_0, \beta_0 \in C_n$ such that

$$\psi(\alpha_0,\beta) \leq \psi(\alpha_0,\beta_0) \leq \psi(\alpha,\beta_0), \ \forall \ \alpha,\beta \in C_n.$$

In particular taking $\alpha = \beta_0$ we see that

$$\psi(\alpha_0,\beta) \leq \psi(\beta_0,\beta_0) \text{ for all } \beta \in C_n.$$

Using the monotonicity of A we obtained that

$$\begin{aligned}
\psi(\alpha,\alpha) &= \sum_{i=1}^{n} \alpha_i \langle x(\alpha) - u_i, v_i - x^* \rangle \\
&= \sum_{j=1}^{n}\sum_{i=1}^{n} \alpha_i\alpha_j \langle u_j - u_i, v_i - x^* \rangle \\
&= \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n} \alpha_i\alpha_j \langle u_j - u_i, v_i - x^* \rangle - \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n} \alpha_i\alpha_j \langle u_j - u_i, v_j - x^* \rangle \\
&= -\frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n} \alpha_i\alpha_j \langle u_j - u_i, v_j - v_i \rangle \leq 0.
\end{aligned}$$

So we have

$$\psi(\alpha_0,\beta) \leq \psi(\alpha,\alpha) \leq 0 \text{ for all } \beta \in C.$$

Taking β with $\beta_j = 0$ for $j \neq i$ and $\beta_i = 1$ for $i = j$ we have

$$\langle x(\alpha_0) - u_i, v_i - x^* \rangle \leq 0, \ \ i = 1, 2, ...n$$

which implies $x(\alpha_0) \in \bigcap_{i=1}^{n} X(u_i, v_i)$.

(b) Since X is reflexive we identify X^{**} with X. Consider

$$A^{-1} \ : \ X^* \ \longrightarrow \ X.$$

Then A^{-1} is monotone and $D(A^{-1}) = R(A)$ which is bounded by assumption, so by (a) above, for each $x \in X$ there exist $x^* \in \overline{CoR(A)}$ such that

$$\langle u - x, v - x^* \rangle \geq 0 \quad \forall \ (u,v) \in A.$$

Remark: Let us now give an example to illustrate that the boundedness condition in the above lemma is crucial. Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = c \ \forall \in \mathbb{R}$. Clearly f is monotone and $D(A) = \mathbb{R}$ which is not bounded. Moreover if we take $y = c - 1$, then we see that there is no $x \in \mathbb{R}$ such that

$$\langle (u - x), (c - (c - 1)) \rangle \geq 0 \text{ for all } u \in \mathbb{R}.$$

i.e there is no $x \in \mathbb{R}$ such that $u \geq x$ for all $u \in \mathbb{R}$. □

Theorem 2.5. *Let X be a real reflexive Banach space and $K \subset X$ be non-empty closed and convex. Assume $A \subset K \times X^*$ is monotone with $0 \in D(A)$ and that $B : K \longrightarrow X^*$ is monotone, hemicontinuous, bounded on bounded subsets and coercive with $x_0 = 0$, then there exist $x \in X$ such that*

$$\langle u - x, v + Bx \rangle \geq 0 \textit{ for all } (u, v) \in A.$$

Proof. We divide the proof into two parts.
(a) Here we assume A is finite. Define $P = coD(A)$ and $coR(A)$. Since A is monotone in $P \times Q$, by Zorns lemma we see that A has a maximal extension $\hat{A}$ on $P \times Q$. let $x \in P$, since $R(\hat{A})$ is bounded then by lemma (2.4) there exist $x^* \in Q$ such that

$$\langle u - x, v - x^* \rangle \geq 0 \text{ for all } (u, v) \in \hat{A}.$$

which implies $(x, x^*) \in \hat{A}$, since $\hat{A}$ is maximal monotone. So we have $D(\hat{A}) = P$. Define

$$A_1 = \{(x, y + Bx) : (x, y) \in \hat{A}\},$$

clearly A_1 is monotone and $D(A_1) = P$, again by lemma (2.4) for each $x^* \in X^*$ there exist $x \in P$ such that

$$\langle u - x, w - x^* \rangle \geq 0 \text{ for all } (u, v) \in A_1.$$

In particular for $x^* = 0$ there exist $x \in P$ such that

$$\langle u - x, w \rangle \geq 0 \text{ for all } (u, v) \in A_1.$$

so we have

$$\langle u - x, v + Bu \rangle \geq 0 \text{ for all } (u, v) \in \hat{A}. \tag{2.1}$$

Now for fixed $(u,v) \in \hat{A}$, define $U_t = x + t(u-x)$, $t \in (0,1)$. Since $D(A_1) = P$ is convex, then we see that there exist $v_t : (u_t, v_t) \in \hat{A}$. So from (2.1) we have

$$0 \leq \langle u_t - x, v_t + Bu_t \rangle = t\langle u - x, v_t + Bu_t \rangle, \; t \in [0,1].$$

Thus we have

$$\langle u - x, v_t + Bu_t \rangle \geq 0 \;\; t \in (0,1]. \tag{2.2}$$

Using the monotonicity of $\hat{A}$ we have

$$0 \leq \langle u - u_t, v - v_t \rangle = (1-t)\langle u - x, v - v_t \rangle, \; t \in [0,1].$$

which implies

$$\langle u - x, v - v_t \rangle \geq 0, \; t \in [0,1). \tag{2.3}$$

(2.2) and (2.3) yields

$$\langle u - x, v + B(x + t(u-x)) \rangle \geq 0.$$

Hemi-continuity of B implies that

$$\langle u - x, v + Bx \rangle \geq 0 \text{ for all } (u,v) \in A.$$

We note that in this proof we did not use the assumption that $0 \in D(A)$.

(b) Let A satisfies the assumptions of the theorem and A_2 be the maximal monotone extension of A in $K \times X^*$. Define

$$E = \{G \in A_2 : G \text{ is finite, monotone and } 0 \in D(G)\},$$

for each $G \in H$ we set

$$B_G = \{(x, Bx) : x \in K, \langle u - x, v + Bx \rangle \geq 0 \; \forall \; (u,v) \in G\}.$$

From part (a) of the proof we see that for each $G \in H$ $B_G \neq \emptyset$. Now let $x \in B_G$ then

$$\begin{aligned} 0 &\leq \langle u - x, v + Bx \rangle \\ &= \langle u, v \rangle + \langle u, Bx \rangle - \langle x, Bx \rangle - \langle x, v \rangle \quad \forall (u,v) \in G, \end{aligned}$$

which implies

$$\langle x, Bx \rangle \leq -\langle x, v \rangle \leq \|x\|\|v\|, \text{ for } v \in G0.$$

Thus we have

$$\frac{\langle x, Bx \rangle}{\|x\|} \leq \|v\|, \text{ for all } x \in D(B_G).$$

Since B is coercive we see that $D(B_G)$ is bounded, which implies that B_G is bounded for each $G \in H$. Since X is reflexive and B_G is bounded then the weak closure $\overline{B_G}^w$ of B_G is weakly compact. Also for each $G_1, G_2 \in H$, $B_{G_1} \cap B_{G_2} = B_{G_1 \cup G_2} \neq \emptyset$. So for any finite number $G_1, G_2 ... G_n$

$$\bigcap_{i=1}^{n} B_{G_i} \neq \emptyset.$$

Thus we have

$$\bigcap_{G \in H} \overline{B_G}^w \neq \emptyset$$

i.e There exist $x_0 \in X,\ x_0^* \in X^*$ such that

$$[x_0, x_0^*] \in \bigcap_{G \in H} \overline{B_G}^w.$$

Since K is closed and convex we see that $\overline{K}^w = K$ and $\overline{B_G}^w \subset K \times X^*$, which implies that $x_0 \in K$. Our next goal is to show that

$$\langle u - x_0, v + Bx_0 \rangle \geq 0 \text{ for all } (u, v) \in A_2.$$

Before that we first show that $(x_0, -x_0^*) \in A_2$. We remarked that $(x, Bx) \in B_G$ if and only if

$$\langle u - x, -v - Bx \rangle \leq 0 \text{ for all } (u, v) \in G. \tag{2.4}$$

For $(x_i, Bx_i) \in H$ $i = 1, 2...n$ and $\lambda_i \geq 0$, $i = 1, 2, ...n$ with $\sum_{i=1}^n \lambda_i = 1$. We set $(x, y) = \sum_{i=1}^n \lambda_i (x_i, Bx_i)$. Using the monotonicity of B and (2.4) we have

$$\begin{aligned} \left\langle u - \sum_{i=1}^{n} \lambda_i x_i, -v - \sum_{j=1}^{n} \lambda_j B x_j \right\rangle &= \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j \langle u - x_i, -v - Bx_j \rangle \\ = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j (\langle u - x_j, -v - Bx_j \rangle &+ \langle x_j - x_i, Bx_i - Bx_j \rangle \\ &+ \langle u - x_i, -v - Bx_i \rangle) \leq 0. \end{aligned}$$

Thus we have

$$\langle u - x, -v - y \rangle \leq 0 \text{ for all } (x, y) \in coB_G \text{ and } [u, v] \in G.$$

Obviously we can extend the above inequality to $\overline{coB_G}$. Since $(x_0, x_0^*) \in \overline{B_G}^w \subset \overline{coB_G}^w = \overline{coB_G}$. We see that

$$\langle u - x_0, -v - x_0^* \rangle \leq 0 \text{ for all } (u, v) \in G.$$

Since A is maximal monotone and $x_0 \in K$ we have $(x_0, -x_0^*) \in A_2$. We choose $w \in A_2 0$ and we define for any $[u, v] \in A_2$ the set $G_0 \in H$ by

$$G_0 = \{(0, w), (u, v), (x_0, -x_0^*)\}.$$

For any $x \in D(B_{G_0})$ we have in particular that

$$\langle u - x_0, v + Bx \rangle \geq 0 \text{ and } \langle x_0 - x, -x_0^* + Bx \rangle \geq 0. \tag{2.5}$$

Set $u_t = x_0 + t(u - x_0)$, $t \in (0, 1)$. Using

$$u_t - x = (1 - t)(x_0 - x) + t(u - x), \ t \in (0, 1),$$

and the monotonicity of B we get

$$0 \leq \langle u_t - x, Bu_t - Bx \rangle = t\langle u - x, Bu_t - Bx \rangle + (1-t)\langle x_0 - x, Bu_t - Bx \rangle.$$

This together with (2.5) gives

$$0 \leq \langle u - x, v + Bu_t \rangle + \langle x_0 - x, Bu_t - x_0^* \rangle.$$

So for $x = x_0$ we have

$$0 \leq \langle u - x_0, v + B(x_0 + t(u - x_0)) \rangle.$$

Letting $t \to 0^+$ and using the Hemi-continuity of B we get the desired result. i.e

$$\langle u - x_0, v + Bx_0 \rangle \geq 0 \text{ for all } (u, v) \in A_2.$$

The proof is complete. □

Theorem 2.6. *Assume X and X^* are reflexive and strictly convex. Let J denote the duality mapping on X and assume that $A \subset X \times X^*$ is monotone, then A is maximal monotone if and only if*

$$R(\lambda J + A) = X^* \tag{2.6}$$

for all $\lambda > 0$ (equivalently for some $\lambda > 0$).

Proof. Suppose (2.6) is satisfied for some $\lambda > 0$. Let $(x_0, y_0) \in X \times X^*$ such that

$$\langle u - x_0, v - y_0 \rangle \geq 0 \text{ for all } (u, v) \in A. \tag{2.7}$$

From the hypothesis we see that there exist $(x_1, y_1) \in A$ such that

$$\lambda J x_1 + y_1 = \lambda J x_0 + y_0. \tag{2.8}$$

Replacing (u, v) by (x_1, y_1) in (2.7) we have

$$\langle x_1 - x_0, \lambda J x_0 - \lambda J x_1 \rangle \geq 0.$$

i.e,

$$\langle x_0 - x_1, \lambda J x_0 - \lambda J x_1 \rangle \leq 0$$

Also by the monotonicity of J we get $\langle x_1 - x_0, Jx_1 - Jx_0\rangle = 0$. Thus we have

$$0 \geq \langle x_1 - x_0, Jx_1 - Jx_0\rangle \geq (\|x_1\| - \|x_0\|)^2 \geq 0.$$

which implies $\|x_1\| = \|x_0\|$ and $\langle x_1, Jx_0\rangle = \|x_1\|^2$, $\langle x_0, Jx_1\rangle = \|x_2\|^2$. The last two equations implies that $Jx_0 \in Jx_1$ and since X^* is strictly convex we have $Jx_0 = Jx_1$. Using the fact that X is reflexive and X^* is strictly convex we obtained that $x_0 = x_1$. This and (2.8) gives $(x_0, x_1) = (x_1, y_1) \in A$.

Now assume A is maximal monotone, with out loss of generality we may assume that $0 \in D(A)$, otherwise we shift the domain of A. Let $x^* \in X^*$ and take $\lambda = 1$, define an operator $\hat{B} :\longrightarrow X^*$ by $\hat{B}x = Jx - x^*$, $x \in X$. We claimed that the operator $\hat{B}$ is

(a) *Monotone*

(b) *Hemi-continuous*

(c) *Bounded on bounded sets*

(d) *Coercive*

Monotonicity of $\hat{B}$ follows directly from that of J. For the hemi-continuity we first note that J is demi-continuous (see proposition 1.11). Let $x_1, x_2, x \in X$, Define a function $g : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$g(\alpha) = \langle x, \hat{B}(x_1 + \alpha x_2)\rangle = \langle x, J(x_1 + \alpha x_2) - x^*\rangle, \ \alpha \in \mathbb{R}.$$

We shall show that g is continuous. Let $\{\alpha_n\}_{n\geq 1} \subset \mathbb{R}$ such that $\alpha_n \to \alpha$. Then from the demi-continuity of J we see that

$$w^*\text{-}lim J(x_1 + \alpha_n x_2) = J(x_1 + \alpha x_2)$$

Since X is reflexive we see that $g(\alpha_n) \to g(\alpha)$, which shows that g is continuous. So we get that $\hat{B}$ is hemi-continuous.

Also the boundedness of $\hat{B}$ follows from that of J (see proposition 1.11). Now

$$\lim_{\|x_n\|\to\infty} \frac{\langle x_n, \hat{Bx_n}\rangle}{\|x_n\|} = \lim_{\|x_n\|\to\infty} \frac{\langle x_n, Jx_n - x^*\rangle}{\|x_n\|} \geq \lim_{\|x_n\|\to\infty} (\|x_n\| - \|x^*\|) = +\infty.$$

Which shows that $\hat{B}$ is coercive. Therefore by Theorem (2.5) there exist $x \in X$ such that

$$\langle u - x, v + Bx\rangle \geq 0, \text{ i.e } \langle u - x, Jx - x^* + v\rangle \geq 0 \text{ for all } (u, v) \in A.$$

Which implies $(x, x^* - Jx) \in A$ since A is maximal monotone. Thus we have $x^* - Jx \in A$ i.e $x^* \in (A+J)(x)$. The proof is complete. $\square$

Corollary 2.7. *Let X be a reflexive Banach space and $A \subset X \times X^*$ be maximal monotone. Assume $B : X \longrightarrow X^*$ is monotone, hemicontinuous and bounded. Then $A+B$ is maximal monotone.*

Proof. Without loss of generality we assume $0 \in D(A)$, this is achieved by shifting the domain of A. Also by Theorem (1.6) we can assume that both X and X^* are strictly convex. Let $x^* \in X^*$, define a new operator $\hat{B} : X \longrightarrow X^*$ by

$$\hat{B}x = Bx + Jx - x^*, \text{ for } x \in X.$$

To see that $\hat{B}$ satisfies the conditions of B in theorem (2.5) we follow the same way as in Theorem (2.6) above. So there exists $x \in X$ such that

$$\langle u - x, Bx + Jx - x^* + v\rangle \geq 0 \text{ for all } (u,v) \in A.$$

Since A is maximal monotone we have $x^* - Bx - Jx \in Ax$ i.e $x^* \in (A+B+J)(x)$, which implies that $X^* \subset R(A+B+J)$. The result follows from Theorem (2.6). $\square$

Theorem 2.8. *Let φ be a proper, convex and l.s.c function on X. Then $\partial\varphi$ is maximal monotone.*

Proof. We first show that $\partial\varphi$ is monotone. Let $x_i^* \in \partial\varphi(x_i)$, $i = 1, 2$. Then we have

$$\varphi(x_1) - \varphi(x_2) \geq \langle x_1 - x_2, x_2^*\rangle \text{ and } \varphi(x_2) - \varphi(x_1) \geq \langle x_2 - x_1, x_1^*\rangle$$

which implies

$$\langle x_1 - x_2, x_1^* - x_2^*\rangle \geq 0.$$

So we have $\partial\varphi$ is monotone.

We now show that $\partial\varphi$ is maximal monotone. By Theorem (2.6) it is enough to show that $R(J + \partial\varphi) = X^*$. Let $x_0^* \in X^*$ be fixed, define a function f by

$$f(x) = \frac{1}{2}\|x\|^2 + \varphi(x) - \langle x, x_0^*\rangle, \ x \in X.$$

Clearly f is proper, convex and l.s.c. Moreover by Theorem (1.26) there exist $x^* \in X^*$ and $c \in \mathbb{R}$ such that $\varphi(x) \geq \langle x, x^*\rangle + c$ for all $x \in X$. With this we easily see that $\lim\limits_{\|x\| \to \infty} f(x) = +\infty$. So by Theorem

(1.25) f has a minimum point, i.e there exists $x_0 \in D(f)$ such that $f(x) - f(x_0) \geq 0$. Equivalently

$$\begin{aligned}\varphi(x) - \varphi(x_0) &\geq \langle x - x_0, x_0^* \rangle + \frac{1}{2}\|x_0\|^2 - \frac{1}{2}\|x\|^2 \\ &\geq \langle x - x_0, x_0^* \rangle - \langle x - x_0, Jx \rangle, \ x \in X,\end{aligned}$$

where we have used the fact that J is the subdifferential of $x \longmapsto \frac{1}{2}\|x\|^2$. For arbitrary $u \in X$ set $x_t = x_0 + t(u - x_0)$, $t \in (0,1)$, then from the above inequality we have

$$\begin{aligned}\varphi(u) - \varphi(x_0) &= \frac{1}{t}(\varphi(x_t) - \varphi(x_0)) \\ &\geq \frac{1}{t}\langle x_t - x_0, x_0^* \rangle - \frac{1}{t}\langle x_t - x_0, Jx_t \rangle \\ &= \langle u - x_0, x_0^* \rangle - \langle u - x_0, Jx_t \rangle.\end{aligned}$$

Observing that J is demi-continuous (see proposition 1.11) we get for $t \to 0^+$

$$\varphi(u) - \varphi(x_0) \geq \langle u - x_0, x_0^* - Jx_0 \rangle \text{ for all } \ u \in X,$$

which proves $x_0^* - Jx_0 \in \partial\varphi(x)$, i.e $x_0^* \in (J + \partial\varphi)(x_0)$. □

2.1.3 Topological Conditions for Maximal Monotone Operators

The next theorem contains a topological conditions that implies maximality of a monotone operator.

Theorem 2.9. *Let X be a reflexive Banach space and $A : X \longrightarrow X^*$ be monotone and hemi-continuous, then A is maximal monotone.*

Proof. Suppose A is not maximal monotone, then there exist $x_0 \in X$ and $y_0 \in X^*$ such that $y_0 \neq x_0$ and

$$\langle x - x_0, Ax - y_0 \rangle \geq 0 \text{ for all } x \in X. \tag{2.9}$$

Set $x_t = tx_0 + (1-t)x$ for $t \in (0,1)$ and $x \in X$. Then $x_t - x_0 = (1-t)(x - x_0)$. Putting this in (2.9) we have

$$0 \leq (1-t)\langle x - x_0, Ax_t - y_0 \rangle \text{ for ll } t \in [0,1].$$

i.e

$$0 \leq \langle x - x_0, A(tx_0 + (1-t)x) - y_0 \rangle \text{ for ll } t \in [0,1).$$

Hemi-continuous of A implies that

$$\langle x - x_0, Ax_0 - y_0\rangle \geq 0 \text{ for ll } x \in X.$$

Thus we have $y_0 = Ax_0$ which contradicts our assumption. Therefore A is maximal monotone. □

We now give an Surjectivity result result for maximal monotone operators

Theorem 2.10. *If $A \subset X \times X^*$ is maximal monotone and coercive. Then $R(A) = X^*$.*

Proof. Without loss of generality we assume that X and X^* are strictly convex. Let $x_0^* \in X^*$, then by Theorem (2.6) for each $\lambda > 0$ there exist $x_\lambda \in D(A)$ and $y_\lambda \in Ax_\lambda$ such that

$$\lambda J x_\lambda + y_\lambda = x_0^* \tag{2.10}$$

Let $x_0 \in X$ such that (2.0) is satisfied then from (2.10) we get

$$\langle x_\lambda - x_0, x_0^*\rangle = \langle x_\lambda - x_0, y_\lambda\rangle + \lambda\|x_\lambda\|^2 - \lambda\langle x_0, Jx_\lambda\rangle.$$

Which implies

$$\begin{aligned}\frac{\langle x_\lambda - x_0, y_\lambda\rangle}{\|x_\lambda\|} + \lambda\|x_\lambda\| &= \frac{\langle x_\lambda - x_0, x_0^*\rangle}{\|x_\lambda\|} + \frac{\lambda\langle x_0, Jx_\lambda\rangle}{\|x_\lambda\|} \\ &\leq \lambda\|x_\lambda\| + \|x_0^*\| + \frac{\|x_0\|\|x_0^*\|}{\|x_\lambda\|}.\end{aligned}$$

and so

$$\frac{\langle x_\lambda - x_0, y_\lambda\rangle}{\|x_\lambda\|} \;\leq\; \|x_0^*\| \;+\; \frac{\|x_0\|\|x_0^*\|}{\|x_\lambda\|}.$$

Coercivity of A implies that $\{x_\lambda\}_{\lambda>0}$ is bounded as $\lambda \to 0^+$. So there exist a subsequence $\{x_{\lambda_n}\}_{n\geq 1} \in X$ and $\hat{x_0} \in X$ such that $x_{\lambda_n} \rightharpoonup \hat{x_0}$. From (2.10) and boundedness $\{x_\lambda\}$ as $\lambda \to 0^+$ and taking $\lambda = \lambda_n$ we see that $y_{\lambda_n} \to x_0^*$. Using the monotonicity of A we have for all $[u, v] \in A$

$$\langle x_{\lambda_n} - u, y_{\lambda_n} - v\rangle \geq 0 \text{ for all } n \geq 1.$$

letting $n \to +\infty$ we have

$$\langle \hat{x_0} - u, x_0^* - v\rangle \geq 0 \text{ for all } (u, v) \in A.$$

Which implies $[\hat{x_0}, x_0^*] \in A$ since A is maximal monotone. Therefore we have $x_0^* \in R(A)$. □

Corollary 2.11. *Let X be a reflexive space. Then the duality mapping is maximal monotone and $R(J) = X^*$*

Proof. It is very easy to see because J is demi-continuous (see proposition 1.11.) which implies hemi-continuity, moreover J is monotone, so by Theorem (2.9) we see that J is maximal monotone. In addition J is coercive, so by Theorem (2.10) we have $R(J) = X^*$ i.e J is surjective.

2.2 The sum of two maximal monotone operators

A problem of great interest because of its application to the existence theory for perturbed partial differential equations is to know whether the sum of two maximal monotone operators is again maximal monotone. Before answering this question let us first establish some facts related to Yosida approximation of maximal monotone operators.

2.2.1 Resolvent and Yosida Approximations of Maximal Monotone Operators

Let us assume that X is reflexive, strictly convex Banach space with strictly convex dual X^*. Let $A \subset X \times X^*$ be maximal monotone, then for all $x \in X$ the inclusion

$$0 \in J(x_\lambda - x) + \lambda A x_\lambda \tag{2.11}$$

has solution $x_\lambda \in X$. We also observe that x_λ is unique. For if there exist $y_\lambda \in X$ such that

$$0 \in J(y_\lambda - x) + \lambda A y_\lambda$$

then we have

$$J(x_\lambda - x) + \lambda y_1 = J(y_\lambda - x) + \lambda y_2 \text{ for some } y_1 \in Ax_\lambda, y_2 \in Ay_\lambda.$$

which implies

$$J(x_\lambda - y_\lambda) = \lambda(y_2 - y_1).$$

From the monotonicity of A we see that

$$\|x_\lambda - y_\lambda\|^2 = \langle x_\lambda - y_\lambda, J(x_\lambda - y_\lambda) = 0,$$

which implies $x_\lambda = y_\lambda$. So x_λ is unique. Define

$$J_\lambda : X \longrightarrow X \text{ by } J_\lambda x = x_\lambda \text{ for all } x \in X,$$

and

$$A_\lambda : X \longrightarrow X^* \text{ by } A_\lambda x = \frac{J(x - x_\lambda)}{\lambda} \text{ for all } x \in X.$$

A_λ is called the *Yosida approximation* of the operator A and it plays an important role in the smooth approximation of A.

2.2.2 Basic Properties of Yosida Approximations

Before we give some basic properties of A_λ and J_λ we first state the following lemma which we shall use in the next result.

Lemma 2.12. *Let X be a reflexive space, and $A \subset X \times X^*$ be maximal monotone. Let $(u_n, v_n) \in A$ such that $u_n \rightharpoonup u$, $v_n \rightharpoonup v$, and either*

$$\limsup \langle u_n - u_m, v_n - v_m \rangle \leq 0 \tag{2.14.}$$

or

$$\limsup \langle u_n - u, v_n - v \rangle \leq 0.$$

Then $[u, v] \in A$.

Proof. Assume $u_n \rightharpoonup u$, and $v_n \rightharpoonup v$. From the monotonicity of A we get

$$\limsup \langle u_n - u_m, v_n - v_m \rangle \geq 0,$$

this together with (2.14) give

$$\lim_{n \to \infty} \langle u_n - u_m, v_n - v_m \rangle = 0.$$

Let $n_k \to +\infty$ be such that $\langle u_{n_k}, v_{n_k} \rangle \to \beta$. (This is possible since $\langle u_n, v_n \rangle$ is bounded in $\mathbb{R}$). Now,

$$\langle u_{n_k} - u_{n_p}, v_{n_k} - v_{n_p} \rangle = \langle u_{n_k}, v_{n_k} \rangle - \langle u_{n_k}, v_{n_p} \rangle - \langle u_{n_p}, v_{n_k} \rangle + \langle u_{n_p}, v_{n_p} \rangle.$$

Using the weak convergence in X(which is reflexive) we have as $k, p \to +\infty$ that

$$2\beta - 2\langle u, v \rangle = \limsup \langle u_{n_k} - u_{m_k}, v_{n_k} - v_{m_k} \rangle \leq 0.$$

i.e $\beta \leq \langle u, v \rangle$. Using the monotonicity of A we have

$$\langle u - x, v - y \rangle \geq \limsup \langle u_n - x, v_n - y \rangle \geq 0 \text{ for all } [x, y] \in A,$$

which implies $(u, v) \in A$ since A is maximal monotone. $\square$

Proposition 2.13. *Let X and Y be strictly convex and reflexive. Then*

(a) A_λ is single valued, monotone, bounded and demi-continuous.
(b) $\|A_\lambda x\| \le |Ax| = \inf\{\|y\| : y \in Ax\}$ for $x \in D(A), \lambda > 0$.
(c) $J_\lambda : X \longrightarrow X$ is bounded on bounded sets and

$$\lim_{\lambda\to 0} J_\lambda x = x, \ \forall \ x \in \overline{coD(A)}.$$

(d) If $\lambda_n \to 0^+$, $x_n \to x$, $A_{\lambda_n} x_n \rightharpoonup y$ and

$$\limsup \langle x_n - x_m, A_{\lambda_n} x_n - A_{\lambda_m} y_m \rangle \le 0, \qquad (2.15.)$$

then $[x,y] \in A$ and $\lim_{n,m\to\infty} \langle x_n - x_m, A_{\lambda_n} x_n - A_{\lambda_m} y_m \rangle = 0$.
(e) For $\lambda \to 0^+$, $A_\lambda x \rightharpoonup A^0 x$ for all $x \in D(A)$, where $A^0 x$ is the element of minimum norm in Ax. If X^* is uniformly convex, then $A_\lambda x \to A^0 x$ for all $x \in D(A)$.

Proof. (a) A_λ is single valued since J is single valued.

Let $x, y \in X$, observing that $A_\lambda x \in AJ_\lambda x$ and using the fact that A and J are monotone we have

$$\begin{aligned}\langle x - y, A_\lambda x - A_\lambda y\rangle &= \langle J_\lambda x - J_\lambda y, A_\lambda x - A_\lambda y\rangle \\ &\quad + \langle (x - J_\lambda x) - (y - J_\lambda y), A_\lambda x - A_\lambda y\rangle \\ &= \langle J_\lambda x - J_\lambda y, A_\lambda x - A_\lambda y\rangle \\ &\quad + \langle (x - x_\lambda) - (y - y_\lambda), J(x - x_\lambda) - J(y - y_\lambda)\rangle \ge 0.\end{aligned}$$

So we have A_λ is monotone.

Let $(u, v) \in A$ be fixed from (2.11) we see that

$$\frac{J(x - x_\lambda)}{\lambda} \in Ax_\lambda.$$

Using the monotonicity of A we obtained that

$$\langle J_\lambda x - u, \frac{1}{\lambda} J(x - x_\lambda) - v\rangle \ge 0,$$

which implies

$$\langle J_\lambda x - u, J(Jx_\lambda - x)\rangle \le \lambda \langle u - J_\lambda x, v\rangle.$$

Thus we have

$$\|J_\lambda x - x\|^2 \le \lambda \|J_\lambda x - x\| \|v\| + \lambda \|u - x\| \|v\| + \|u - x\| \|J_\lambda x - x\|.$$

Which shows that J_λ and A_λ are bounded.

Let $x_n \to x_0$, set $J_\lambda x_n = u_n$, $A_\lambda x_n = v_n$. Then from the equation

$$J(u_n - x_n) + \lambda v_n = 0,$$

it follows that

$$\langle (u_n - x_n) - (u_m - x_m), J(u_n - x_n) - J(u_m - x_m) \rangle$$
$$+ \lambda \langle u_n - u_m, v_n - v_m \rangle + \lambda \langle x_m - x_n, v_n - v_m \rangle = 0.$$

Since as seen before J_λ is bounded we get

$$\limsup \langle u_n - u_m, v_n - v_m \rangle \leq 0$$

and

$$\limsup \langle (u_n - x_n) - (u_m - x_m), J(u_n - x_n) - J(u_m - x_m) \rangle \leq 0.$$

Let $n_k \to +\infty$ such that $u_{n_k} \rightharpoonup u$, $v_{n_k} \rightharpoonup v$, and $J(u_{n_k} - x_{n_k}) \rightharpoonup w$. Then by lemma (2.12) we have $(u, v) \in A$ and $(u - x_0, w) \in J$. Therefore

$$J(u - x_0) + \lambda v = 0.$$

Which implies $u = J_\lambda x_0$, $v = A_\lambda x_0$ and by the uniqueness of limit we have $J_\lambda x_n \rightharpoonup u$ and $A_\lambda x_n \rightharpoonup v$.

(b) Let $(x, x^*) \in A$ and $\lambda > 0$. Then from the monotonicity of A we have

$$\begin{aligned} 0 &\leq \langle x - J_\lambda x, x^* - A_\lambda x \rangle \\ &= \langle x - J_\lambda x, x^* \rangle - \lambda^{-1} \langle x - J_\lambda x, J(x - J_\lambda x) \rangle \\ &\leq \|x - J_\lambda x\| \|x^*\| - \lambda^{-1} \|x - J_\lambda x\|^2 \end{aligned}$$

which implies

$$\lambda^{-1} \|x - J_\lambda x\|^2 \leq \|x - J_\lambda x\| \|x^*\|,$$

i.e

$$\|A_\lambda x\| \leq \|x^*\| \text{ for all } x^* \in Ax.$$

Therefore we have

$$\|A_\lambda x\| \leq |Ax| = \inf\{\|x^*\| : x^* \in Ax\}.$$

(c) Let $x \in \overline{coD(A)}$ and $(u, v) \in A$. Then by the monotonicity of A we get

$$\begin{aligned} 0 &\leq \langle J_\lambda x - u, A_\lambda x - v \rangle \\ &= \langle J_\lambda x - u, A_\lambda x \rangle + \langle u - J_\lambda x, v \rangle \\ &= \langle J_\lambda x - x, A_\lambda x \rangle + \langle x - u, A_\lambda x \rangle + \langle u - J_\lambda x, v \rangle. \end{aligned}$$

So we have

$$\|J_\lambda x - x\|^2 \le \langle x-u, J(x-J_\lambda x)\rangle + \lambda\langle u - J_\lambda x, v\rangle.$$

Let $\lambda_n \to 0^+$ such that $J(J_{\lambda_n}x - x) \rightharpoonup y$. Then we have

$$\limsup \|J_\lambda x - x\|^2 \le \langle x-u, y\rangle.$$

The inequality above can be extended to all $u \in \overline{coD(A)}$. In particular taking $u = x$ we have the result, i.e $J_\lambda x \to x$ as $\lambda \to 0^+$.

(d) To show that $(x,y) \in A$ we shall show that $J_{\lambda_n}x_n \to x$ and

$$\lim_{m,n\to\infty} \langle J_{\lambda_n}x_n - J_{\lambda_m}x_m, A_{\lambda_n}x_n - A_{\lambda_m}x_m\rangle = 0,$$

so that we can apply lemma (2.12) since

$$(J_{\lambda_n}x_n, A_{\lambda_n}x_n) \in A \text{ for all } n \ge 1.$$

Now

$$\begin{aligned}\langle x_n - x_m, &A_{\lambda_n}x_n - A_{\lambda_m}x_m\rangle = \langle x_n - x_m, AJ_{\lambda_n}x_n - AJ_{\lambda_m}x_m\rangle \\ &= \langle J_{\lambda_n}x_n - J_{\lambda_m}x_m, AJ_{\lambda_n}x_n - AJ_{\lambda_m}x_m\rangle + \langle (x_n - J_{\lambda_n}x_n) \\ &- (x_m - J_{\lambda_m}x_m), A_{\lambda_n}x_n - A_{\lambda_m}x_m\rangle \\ &\ge \langle (x_n - J_{\lambda_n}x_n) - (x_m - J_{\lambda_m}x_m), J(x_n - J_{\lambda_n}x_n)\lambda_n^{-1} \\ &- J(x_m - J_{\lambda_m}x_m)\lambda_m^{-1}\rangle.\end{aligned}$$

Since x_n and $A_{\lambda_n}x_n$ are bounded on bounded sets of X andX^* respectively we see that

$$\lim_{m,n\to\infty} \langle x_n - x_m, A_{\lambda_n}x_n - A_{\lambda_m}x_m\rangle = 0$$

and

$$\lim_{m,n\to\infty} \langle J_{\lambda_n}x_n - J_{\lambda_m}x_m, A_{\lambda_n}x_n - A_{\lambda_m}x_m\rangle = 0.$$

Also we have

$$\lim_{n\to\infty} (J_{\lambda_n}x_n - x) = \lim_{n\to\infty} \lambda_n J^{-1}(A_{\lambda_n}x_n) = 0.$$

Thus we have $J_{\lambda_n}x_n \to x$ and since $A_{\lambda_n}x_n \rightharpoonup y$ applying lemma (2.12) we see that $(x,y) \in A$.

(e) Since Ax is a closed, convex subset of X^*, and X^* is reflexive and strictly convex, then the projection A^0x of 0 onto Ax is well defined and unique.

Let $x \in D(A)$ and $\lambda_n \to 0$ such that $A_{\lambda_n}x \rightharpoonup y$, we have seen in the above proof that $y \in Ax$. Now since $\|A_{\lambda_n}x\| \le \|A^0x\|$ as seen above and $\|y\| \le \liminf \|A_{\lambda_n}x\| \le \|A^0x\|$, then we have $\|y\| \le \|A^0x\|$. Also we have that $\|A^0x\| \le \|y\|$. Thus $\|A^0x\| = \|y\|$. Since A^0x is unique we see that $y = A^0x$, and therefore $\|A_{\lambda_n}x\| \rightharpoonup A^0x$. □

Proposition 2.14 *If X is a Hilbert space, then*
(a) $J_\lambda = (I + \lambda A)^{-1}$ is non expansive in X, i.e

$$\|J_\lambda x - J_\lambda y\| \leq \|x - y\| \textit{ for all } x, y \in X.$$

(b) A_λ is Lipschitz with Lipschitz constant λ^{-1}, i.e

$$\|A_\lambda x - A_\lambda y\| \leq \lambda^{-1}\|x - y\| \textit{ for all } x, y \in D(A).$$

Proof. By definition and our assumption that X is a Hilbert space we have for each $x, y \in D(A)$

$$\lambda^{-1}(x - J_\lambda x) = \lambda^{-1}J(x - J_\lambda x) \in AJ_\lambda x,$$

and

$$\lambda^{-1}(y - J_\lambda y) = \lambda^{-1}J(y - J_\lambda y) \in AJ_\lambda y.$$

Using the monotonicity of A we get

$$\begin{aligned}0 &\leq \langle J_\lambda x - J_\lambda y, \lambda^{-1}J(x - J_\lambda x) - \lambda^{-1}J(y - J_\lambda y)\rangle \\ &= \langle J_\lambda x - J_\lambda y, \lambda^{-1}(x - y) - \lambda^{-1}(J_\lambda x - J_\lambda y),\end{aligned}$$

which implies that

$$\begin{aligned}\langle J_\lambda x - J_\lambda y, J_\lambda x - J_\lambda y\rangle &\leq \langle J_\lambda x - J_\lambda y, x - y\rangle \\ &\leq \|J_\lambda x - J_\lambda y\|\|x - y\|.\end{aligned}$$

Therefore we have

$$\|J_\lambda x - J_\lambda y\| \leq \|x - y\|, \text{ for all } x, y \in D(A).$$

(b) For each $x, y \in D(A)$ we have

$$A_\lambda x = \lambda^{-1}J(x - x_\lambda) = \lambda^{-1}(x - x_\lambda),$$

and

$$A_\lambda y = \lambda^{-1}J(y - y_\lambda) = \lambda^{-1}(y - y_\lambda).$$

Now

$$\begin{aligned}\|A_\lambda x - A_\lambda y\|^2 &= \langle A_\lambda x - A_\lambda y, A_\lambda x - A_\lambda y\rangle \\ &= \langle A_\lambda x - A_\lambda y, \lambda^{-1}(x - x_\lambda) - \lambda^{-1}(y - y_\lambda)\rangle \\ &= \langle A_\lambda x - A_\lambda y, \lambda^{-1}(x - y)\rangle - \langle A_\lambda x - A_\lambda y, \lambda^{-1}(x_\lambda - y_\lambda)\rangle \\ &\leq \lambda^{-1}\langle A_\lambda x - A_\lambda y, x - y\rangle \\ &\leq \lambda^{-1}\|A_\lambda x - A_\lambda y\|\|x - y\|.\end{aligned}$$

Thus we have

$$\|A_\lambda x - A_\lambda y\| \leq \lambda^{-1}\|x - y\|, \text{ for all } x, y \in X.$$

□

Corollary 2.15. *Let be a reflexive Banach space and let A be a maximal monotone in $X \times X^*$. Then $\overline{D(A)}$ and $\overline{R(A)}$ are convex.*

Proof. For each $x \in \overline{coD(A)}$ we have $J_\lambda x \to x$ as $\lambda \to 0^+$ and $J_\lambda x \in D(A)$ for all $\lambda > 0$. Therefore we have $\overline{D(A)} = \overline{coD(A)}$. Hence $\overline{D(A)}$ is convex since $\overline{coD(A)}$ is convex.

Also we have $A^{-1} : R(A) \longrightarrow X$ is maximal monotone. So following the same argument as above we see that $\overline{R(A)}$ is convex.

Lemma 2.16. *Let $\{x_n\} \subset X, \{y_n\} \subset X^*$ such that $x_n \to 0$ and $\|y_n\| \to +\infty$ as $n \to \infty$. Let $B(0, r)$ denote the closed ball with radius r in X. Then for all $r > 0$ there exist $x_0 \in B(0, r)$ and subsequences $\{x_{n_k}\} \subset X, \{y_{n_k}\} \subset X$ such that*

$$\lim_{n\to\infty} \langle x_{n_k} - x_0, y_{n_k}\rangle = -\infty.$$

Theorem 2.17 *Let A be monotone subset of $X \times X^*$. Then A is locally bounded in any interior point of $D(A)$.*

Proof. Let $x_0 \in D(A)$, without loss of generality we may assume that $x_0 = 0$ (This can be achieved by shifting the domain of A). Let us assume that A is not locally bounded at 0, then there exist $(x_n, y_n) \in A$ such that $\|x_n\| \to 0$ and $\|y_n\| \to +\infty$. So applying lemma (2.16) we see that for all $r > 0$ there exist $x_1 \in B(0, r)$ and $\{x_{n_k}\} \subset X, \{y_{n_k}\} \subset X$ such that

$$\lim_{n\to\infty} \langle x_{n_k} - x_1, y_{n_k}\rangle = -\infty. \tag{2.16}$$

For r sufficiently small we see that $x_1 \in D(A)$. So by the monotonicity of A we have

$$\langle x_{n_k} - x_1, y_{n_k}\rangle \geq 0,$$

which contradicts (2.16). Hence we have the result.

Theorem 2.18. *Let X be a reflexive Banach space. Let A and B be two monotone subsets of $X \times X^*$ such that $intD(A) \cap D(B) \neq \emptyset$. Then $A + B$ is maximal monotone.*

Proof. Without loss of generality we may assume that X and X^* are strictly convex. Moreover, we also assumed that $0 \in intD(A) \cap D(B)$, $0 \in A0$ and $0 \in B0$. (This can be achieved by shifting the domains and ranges of A and B respectively).

We shall prove that $R(A+B+J) = X^*$. Let $y \in X^*$, since the operator B_λ is monotone, demi-continuous and bounded. Then by corollary (2.7) and theorem (2.6) we see that for each $\lambda > 0$ the inclusion

$$y \in Jx_\lambda + Ax_\lambda + B_\lambda x_\lambda \tag{2.17}$$

has a unique solution $x_\lambda \in D(A)$. Now using (2.17) and the fact that

$$\langle x_\lambda, Ax_\lambda \rangle \geq 0 \text{ and } \langle x_\lambda, B_\lambda x_\lambda \rangle \geq 0$$

(this is true because $(0,0) \in A$ and $(0,0) \in B$) we get

$$\begin{aligned}\langle x_\lambda, y \rangle &= \langle x_\lambda, Jx_\lambda + Ax_\lambda + B_\lambda x_\lambda \rangle \\ &= \langle x_\lambda, Ax_\lambda \rangle + \langle x_\lambda, B_\lambda x_\lambda \rangle + \langle x_\lambda, Jx_\lambda \rangle \\ &\geq \langle x_\lambda, Jx_\lambda \rangle.\end{aligned}$$

Thus we have $\|x_\lambda\| \leq \|y\|$.

Also since $0 \in intD(A)$ it follows from theorem (2.17) that there exists $M > 0$, $\rho > 0$ such that

$$\|x^*\| \leq M \text{ for all } x^* \in Ax, \ \|x\| \leq \rho. \tag{2.18}$$

Now using (2.17) and the monotonicity of A we get for each $\omega \in D(A)$ such that $\|\omega\| = 1$

$$\begin{aligned}\langle x_\lambda - \rho\omega, y \rangle &= \langle x_\lambda - \rho\omega, Jx_\lambda + Ax_\lambda + B_\lambda x_\lambda \rangle \\ &= \langle x_\lambda - \rho\omega, Ax_\lambda - A(\rho\omega) \rangle + \langle x_\lambda - \rho\omega, A(\rho\omega) \rangle \\ &+ \langle x_\lambda - \rho\omega, Jx_\lambda + B_\lambda x_\lambda \rangle \\ &\geq \langle x_\lambda - \rho\omega, A(\rho\omega) \rangle + \langle x_\lambda - \rho\omega, Jx_\lambda + B_\lambda x_\lambda \rangle.\end{aligned}$$

Thus we have

$$\langle x_\lambda - \rho\omega, Jx_\lambda + B_\lambda x_\lambda - y \rangle + \langle x_\lambda - \rho\omega, A(\rho\omega) \rangle \leq 0.$$

Using the inequality in (2.18) we have

$$\|x_\lambda\|^2 - \rho\langle \omega, B_\lambda x_\lambda \rangle \leq M(\|x_\lambda\| + \rho) + \|x_\lambda\|(\|y\| + \rho) + \rho\|y\|.$$

Hence,

$$\|x_\lambda\|^2 + \rho\|B_\lambda x_\lambda\| \leq \|x_\lambda\|(\rho + M + \|y\|) + M_\rho, \ \rho > 0.$$

We may therefore conclude that $\{B_\lambda x_\lambda\}$, $y_\lambda = y - Jx_\lambda - B_\lambda x_\lambda$ are bounded. Since X^* is reflexive, without loss of generality we may assume that $x_\lambda \rightharpoonup x_0$, $B_\lambda x_\lambda \rightharpoonup y_1$, $y_\lambda \rightharpoonup y_2$ and $Jx_\lambda \rightharpoonup y_0$.

Now for each $\beta, \mu > 0$ we have

$$y \in Jx_\lambda + Ax_\lambda + B_\lambda x_\lambda \text{ and } y \in Jx_\mu + Ax_\mu + B_\mu x_\mu,$$

which implies

$$0 \in Ax_\lambda - Ax_\mu + B_\lambda x_\lambda - B_\mu x_\mu + Jx_\lambda - Jx_\mu.$$

Since $A + J$ is monotone we have

$$\begin{aligned}\langle x_\lambda - x_\mu, B_\lambda x_\lambda - B_\mu x_\mu\rangle = \langle x_\lambda - x_\mu, Ax_\lambda - Ax_\mu\rangle \\ + \langle x_\lambda - x_\mu, Jx_\lambda - Jx_\mu\rangle \leq 0.\end{aligned}$$

By proposition (2.13)(d) we have

$$\lim_{\beta,\mu \to 0} \langle x_\lambda - x_\mu, B_\lambda x_\lambda - B_\mu x_\mu\rangle = 0, \text{ and } [x, y] \in B.$$

Also from (2.17) we have

$$\lim_{\beta,\mu \to 0} \langle x_\lambda - x_\mu, (Jx_\lambda + y_\lambda) - (Jx_\mu + y_\mu)\rangle = 0, \; y_\lambda \in Ax_\lambda, y_\mu \in Ax_\mu.$$

Therefore by lemma (2.12) we have that $[x_0, y_0 + y_2] \in A + J$. Now letting $\lambda \to 0^+$ we see that

$$y \in Ax_0 + Jx_0 + Bx_0.$$

The proof is complete. □

Chapter 3

On the Characterization of Maximal Monotone Operators

In this chapter we present a short proof for the Rockafellar's characterization of maximal monotone operators (Theorem 2.6) in Banach space through convex analysis approach following C.Simons snd C.Zalinescu. As a consequence we get a generalization of Theorem 2.6. Furthermore some application of the monotone operators theory to the solvability of nonlinear Partial Differential Equation, will be given.

3.1 Rockafellar's characterization of maximal monotone operators.

Throughout this section X will denote a real reflexive Banach space with its dual X^*. Therefore the dual of $X \times X^*$ is canonically isomorphic to $X^* \times X$ and as usual we define its duality pairing for $(x, x^*) \in X \times X^*$ and $(u^*, u) \in X^* \times X$ by

$$\langle (x, x^*), (u^*, u) \rangle = \langle x, u^* \rangle + \langle u, x^* \rangle.$$

Theorem 3.1. *Let X be a reflexive Banach space. Let $A \subset X \times X^*$ be monotone. Then A is maximal monotone if and if*

$$grA + gr(-J_X) = X \times X^*. \tag{3.1}$$

Where J_X is the duality mapping on X.

For a proof of this theorem we need the following lemmas.

Lemma 3.2. *Let $A \subset X \times X^*$ be a maximal monotone operator and $(y, y^*) \in X \times X^*$, then*

$$\inf_{(a,a^*)\in A} \langle a - y, a^* - y^* \rangle \leq 0 \tag{3.2}$$

$$\text{with equality } \Leftrightarrow (y, y^*) \in A. \tag{3.3}$$

Proof. Let $(y, y^*) \in X \times X^*$. If $(y, y^*) \in A$, then it is not difficult to see by monotonicity A that

$$\inf_{(a,a^*)\in A} \langle a - y, a^* - y^* \rangle = 0.$$

Otherwise if $(y, y^*) \notin A$ then by maximality of A (see Definition 2.1.) there exists $(u, u^*) \in X \times X^*$ such that $\langle u - y, u^* - y^* \rangle < 0$. It follows that the inequality in (3.2.) holds.
For the second part of the Lemma, it remains only to show that given $(y, y^*) \in X \times X^*$ such that $\inf_{(a,a^*)\in A} \langle a - y, a^* - y^* \rangle = 0$, then $(y, y^*) \in A$. Indeed if $\inf_{(a,a^*)\in A} \langle a - y, a^* - y^* \rangle = 0$, then we have

$$\langle a - y, a^* - y^* \rangle = 0 \text{ for all } (a, a^*) \in A$$

and since A is maximal monotone we have that $(y, y^*) \in A$. The proof is complete. □

Besides for later use let us consider the following function $g : X \times X^* \longrightarrow \overline{\mathbb{R}}$ defined by

$$g(y, y^*) = \sup_{(a,a^*)\in A} [\langle a, y^* \rangle + \langle y, a^* \rangle - \langle a, a^* \rangle]. \tag{3.4}$$

Then (3.2) can be written as

$$\forall\ (y, y^*) \in X \times X^*,\ \langle y, y^* \rangle \leq g(y, y^*) \tag{3.5}$$

$$\text{with equality } \Leftrightarrow (y, y^*) \in A. \tag{3.6}$$

This is easily seen because from (3.2) we have

$$\begin{aligned} 0 \geq \inf_{(a,a^*)\in A} \langle a - y, a^* - y^* \rangle &= -\sup_{(a,a^*)\in A} [\langle y - a, a^* - y^* \rangle] \\ &= -\sup_{(a,a^*)\in A} [\langle a, y^* \rangle + \langle y, a^* \rangle - \langle a, a^* \rangle] - \langle y, y^* \rangle] \\ &= \langle y, y^* \rangle - g(y, y^*). \end{aligned}$$

Thus we have $\langle y, y^* \rangle \leq g(y, y^*)$.

The function g is a convex function as a point-wise supremum of a family of convex functions. Taking $(y, y^*) \in A$ we have $g(y, y^*) = \langle y, y^* \rangle$, which shows that g is proper. Also g is lower semicontinuous as a point-wise supremum of bounded linear maps.

Lemma 3.3. *Let $A \subset X \times X^*$ be maximal monotone and $(u^*, u) \in \partial g(v, v^*)$. Then $\langle v - u, v^* - u^* \rangle \leq 0$. Moreover, if*

$$\langle v - u, v^* - u^* \rangle = 0, \text{ then } (u, u^*) \in A.$$

Proof. From (3.4) we have that

$$\begin{aligned}\langle v - u, v^* - u^* \rangle &= \langle v, v^* \rangle - \langle v, u^* \rangle - \langle u, v^* \rangle + \langle u, u^* \rangle \\ &\leq g(v, v^*) - \langle v, u^* \rangle - \langle u, v^* \rangle + \langle u, u^* \rangle\end{aligned}$$

Now let $(a, a^*) \in A$ be fixed, using (3.5) and the assumption that $(u, u^*) \in \partial g(v, v^*)$ we have

$$\begin{aligned}g(v, v^*) &\leq g(a, a^*) - \langle (a, a^*) - (v, v^*), (u^*, u) \rangle \\ &\leq \langle a, a^* \rangle + \langle v - a, u^* \rangle + \langle u, v^* - a^* \rangle.\end{aligned}$$

Thus we have

$$\langle v - u, v^* - u^* \rangle \leq \langle a, a^* \rangle - \langle a, u^* \rangle - \langle u, a^* \rangle + \langle u, u^* \rangle = \langle a - u, a^* - u^* \rangle$$

Hence

$$\langle v - u, v^* - u^* \rangle \leq \inf_{(a,a^*) \in A} \langle a - u, a^* - u^* \rangle. \tag{3.7}$$

Using (3.2) we see that $\langle v - u, v^* - u^* \rangle \leq 0$. Now if $\langle v - u, v^* - u^* \rangle = 0$ then (3.7) gives

$$\inf_{(a,a^*) \in A} \langle a - u, a^* - u^* \rangle \geq 0$$

which implies $(u, u^*) \in A$ since A is maximal monotone. □

Lemma 3.4. *Let X be a reflexive Banach space and $X \times X^*$ be endowed with the euclidean norm. Denote by $J_{X \times X^*}$ the duality mapping on $X \times X^*$. Then*

$$J_{X \times X^*}(x, x^*) = J_X(x) \times J_{X^*}(x^*) \text{ for all } (x, x^*) \in X \times X^*.$$

Proof. Using the fact that $X \times X^*$ with the euclidean norm is strictly convex, it is enough to show that

$$J_X(x) \times J_{X^*}(x^*) \in J_{X \times X^*}(x, x^*) \text{ for each } (x, x^*) \in X \times X^*.$$

Let $(y^*, y^{**}) \in J_X(x) \times J_{X^*}(x^*)$, then we have

$$\langle x, y^* \rangle = \|x\|^2, \ \|x\| = \|y^*\|^2, \text{ and } \langle x^*, y^{**} \rangle = \|x^*\|^2, \ \|x^*\| = \|y^{**}\|.$$

Now

$$\langle (x, x^*), (y^*, y^{**}) \rangle = \langle x, y^* \rangle + \langle x^*, y^{**} \rangle = \|x\|^2 + \|x^*\|^2 = \|(x, x^*)\|^2.$$

Also we have

$$\|(x, x^*)\|^2 = \|x\|^2 + \|x^*\|^2 = \|y^*\|^2 + \|y^{**}\|^2 = \|(y^*, y^{**})\|^2.$$

Therefore we have

$$J_X(x) \times J_{X^*}(x^*) \in J_{X \times X^*}(x, x^*),$$

which implies

$$J_X(x) \times J_{X^*}(x^*) = J_{X \times X^*}(x, x^*).$$

We are now ready to prove Theorem 3.1.

Proof of theorem 3.1. Assume (3.1) holds, let $(y, y^*) \in X \times X^*$ be such that $\langle a - y, a^* - y^* \rangle \geq 0$ for all $(a, a^*) \in grA$. By assumption there exists $(a, a^*) \in grA$ and $(u, u^*) \in gr(-J_X)$ such that $(y, y^*) = (a, a^*) + (u, u^*)$. Then we have

$$0 \leq \langle a - y, a^* - y^* \rangle = \langle -u, -u^* \rangle = -\langle u, -u^* \rangle = -\|u\|^2 = -\| - u^*\|^2.$$

Thus we have $u = 0$ and $u^* = 0$. It follows that $(y, y^*) = (a, a^*) \in grA$. Hence A is maximal monotone. Conversely, assume A is maximal monotone, without loss of generality we shall prove that $(0, 0) \in grA + gr(-J_X)$. Consider the function g defined in (3.5). Define a new function $h : X \times X^* \longrightarrow \overline{\mathbb{R}}$ by

$$h(x, x^*) = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 + g(x, x^*).$$

Clearly g is proper, lower semicontinuous, coercive and convex. Since $X \times X^*$ is reflexive, then by Theorem (1.25) there exists a minimizer $(v, v^*) \in X \times X^*$ of h on $X \times X^*$. Hence $(0, 0) \in \partial h(v, v^*)$. Moreover using the idea of Example 1 in Section 2 of Chapter 1 we see that

$$(0, 0) \in J_X(v) \times J_X^{-1}(v^*) + \partial g(v, v^*).$$

So there exists $(u, u^*) \in \partial g(v, v^*)$ such that $-u^* \in J_X(v)$ and $-u \in J_X^{-1}(v^*)$ (equivalently $-v^* \in J_X(u)$). Using Lemma (3.2) we have $\langle v - u, v^* - u^* \rangle \leq 0$. We also have

$$\langle v, -u^* \rangle = \|v\|^2 = \|u^*\|^2, \text{ and } \langle u, v^* \rangle = \|u\|^2 = \|v^*\|^2,$$

and so

$$0 \geq \langle v-u, v^*-u^* \rangle = \langle v, v^* \rangle + \langle v, -u^* \rangle + \langle u, -v^* \rangle + \langle u, u^* \rangle$$
$$\geq \|v\|^2 - 2\|u\|\|v\| + \|u\|^2 = (\|v\| - \|u\|)^2 \geq 0.$$

Hence $\langle v-u, v^*-u^* \rangle = 0$, and $\|v\| = \|u\|$. Using the second part of Lemma (1.1) we have $(u, u^*) \in grA$. Now

$$0 = \langle v-u, v^*-u^* \rangle = \langle -u, u^* \rangle + \langle u, v^* \rangle + \langle v - v^* \rangle + \langle u, u^* \rangle$$
$$= \|v\|^2 + \|u\|^2 - \|v\|^2 + \langle u, u^* \rangle$$

which implies that $\langle -u, u^* \rangle = -\|u\|^2$ i.e $u^* \in J_X(-u)$. Hence $(-u, -u^*) \in gr(-J_X)$, Since $(u, u^*) \in grA$, we deduce that

$$(0,0) = (u, u^*) + (-u, -u^*) \in grA + gr(-J_X).$$

The proof is complete. □

Theorem 3.5. *Let X be a reflexive Banach space. Let $A \subset X \times X^*$ be monotone. If A is maximal monotone, then $A + J_X$ is onto. Conversely, if $A + J_X$ is onto and both J_X and J_X^{-1} are single valued, then A is maximal monotone.*

Proof. Assume A is maximal monotone. Let $y^* \in X^*$ then we see that $(0, y) \in X \times X^*$. According to Theorem (3.4),

$$(0, y^*) \in grA + gr(-J_X).$$

So there exists $(a, a^*) \in A$, $(u, u^*) \in -J_X$ such that

$$(a + u, a^* + u^*) = (0, y^*),$$

which implies that $a + u = 0$ and $a^* + u^* = y^*$. So we have $a = -u$ and $y^* \in A(a) + J_X(a)$ (here we have used the fact that $u^* \in -J_X(u)$). Thus we have $y^* \in (A + J)(a)$ which implies $X^* \subset A + J_X$. Hence

$$R(A + J_X) = X^*.$$

For the converse of the proof, one can follow the same process as in theorem (2.6).

Chapter 4

Applications

In this part we present some examples of monotone operators and of uniformly monotone operators (including the opposite of the Laplacian $-\Delta$ with domain contains in $H_0^1(\Omega)$, where Ω is an open, bounded subset of $\mathbb{R}^n$, and it turns out to be maximal monotone and coercive) which arise in weak formulations of Nonlinear Elliptic problems following Adam Besenyei.

4.1 Laplacian

In this section we shall prove that $-\Delta$ is maximal monotone and coercive, thus by theorem (2.10) it is surjective. First consider

$$-\Delta : D(\Delta) \subset H_0^1(\Omega) \longrightarrow L^2(\Omega)$$

where Ω is an open bounded subset of $\mathbb{R}^n$ and in fact

$$D(\Delta) \ = \ H_0^1(\Omega) \cap H^2(\Omega)\,.$$

Then for any $u \in D(\Delta)$, using Green's formula we have

$$\langle -\Delta u, u\rangle \ = \ \int_\Omega |\nabla u|^2 \ \geq \ 0,$$

so $-\Delta$ is monotone.

Now to show that $-\Delta$ is maximal monotone, by Theorem (2.6), it suffices to show that

$$R(-\Delta + I) = L^2(\Omega).$$

Consider the following Dirichlet boundary value problem.

$$\begin{cases} -\Delta u + u &= f \text{ in } \Omega, \\ \qquad\quad u &= 0 \text{ on } \partial\Omega. \end{cases}$$

Using Theorem (1.25) or Lax Milgram Theorem (see [11]) we see that the above Dirichlet problem has a unique solution. So according to Theorem (2.6) we have that $-\Delta$ is maximal monotone.

Now to conclude indeed that $-\Delta$ is surjective we just need to show that it is coercive, and the result will follow from Theorem (2.10). But we have

$$\frac{\langle -\Delta u, u\rangle}{\|u\|_{H_0^1}} = \frac{\int_\Omega |\nabla u|^2}{\|u\|_{H_0^1}} = \frac{\|\nabla u\|^2_{L^2(\Omega)}}{\|u\|_{H_0^1}} \geq \frac{C\|u\|^2_{H_0^1}}{\|u\|_{H_0^1}} = C\|u\|_{H_0^1}$$

where C is a positive constant (independent from u) by Poincaré inequality.

Thus we have

$$\lim_{\|u\|_{H_0^1}\to\infty} \frac{\langle -\Delta u, u\rangle}{\|u\|_{H_0^1}} = +\infty.$$

Therefore, by Theorem (2.10) we see that $-\Delta$ is surjective (in agreement with the classical result of existence existence of solutions (in fact unique) to Poisson equations with homogeneous Dirichlet boundary condition on bounded domains.

Remark. Similar results hold for the p-Laplacian (See [13]).

4.2 Uniformly Monotone Operators

Let X be a normed space and X^* denotes its dual. Then an operator $A : X \longrightarrow X^*$ is called *uniformly monotone* if there exist $p \geq 2, \gamma > 0$ such that

$$\langle A(u_1) - A(u_2), u_1 - u_2\rangle \geq \gamma\|u_1 - u_2\|^p_X, \text{ for all } u_1, u_2 \in X. \quad (4.1)$$

It is obvious that a uniformly monotone operator is a monotone operator, and an immediate example of uniformly monotone operator is the duality mapping in Hilbert space with $p = 2$ and $\gamma = 1$.

In what follows we study operators which are obtained by considering the weak formulation of an Elliptic equation with some boundary conditions.

Let X be a linear subspace of $W^{1,p}(\Omega)$ where $\Omega \subset \mathbb{R}^n$ is bounded (with sufficiently smooth boundary), $p \geq 2$. Define an operator $A : X \longrightarrow X^*$ by

$$\langle A(u), v\rangle = \int_\Omega \left(\sum_{i=1}^{n} a_i(x, u, \nabla u) D_i v + a_0(x, u, \nabla u) v \right) dx \qquad (4.2)$$

where $a_i(x, u, \nabla u) = a_i(x, u(x), \nabla u(x))$ and D_i denotes the distributional derivative with respect to the $i - th$ variable.

Consider the abstract Equation

$$A(u) = f, \qquad (4.3)$$

where $f \in X^*$(which may be obtained as a weak formulation of an elliptic boundary value problem). Supposing the Uniform monotonicity of A (and some other properties) of an operator of the form (4.2) one can prove the existence and uniqueness of solution to the above abstract equation.
An example is giving by the operator corresponding to

$$a_i(x, \xi) = \xi_i \|\xi\|^{p-2} \quad (i = 1, 2, 3...n),$$
$$a_0(x, \xi) = \xi_0 \|\xi\|^{p-2}$$

where $\xi = (\xi_0, \xi_1, \xi_2, ...\xi_n)$.
For instance Lions [10] proved that the following three conditions are sufficient for the existence of a solution to (4.3).
(C1) The functions $a_i : \Omega \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$, $(i = 1, 2, 3...n)$ are of Caratheodory type. i.e for all $\xi \in \mathbb{R}^{n+1}$, $x \mapsto a_i(x, \xi)$ is measurable and for $a.e.$ $x \in \Omega$ $\xi \mapsto a_i(x, \xi)$ is continuous.

(C2) There exists a constant $c > 0$, and a function $k \in L^q(\Omega)$ such that for $a.e.$ $x \in \Omega$ and for all $\xi \in \mathbb{R}^{n+1}$

$$|a_i(x, \xi)| \leq c\|\xi\|^{p-1} + k(x).$$

(C3) There exists a constant $C > 0$ such that for $a.e$ $x \in \Omega$ and for all $\tilde{\xi}, \xi \in \mathbb{R}^{n+1}$

$$\sum_{i=1}^{n} (a_i(x, \xi) - a_i(x, \tilde{\xi}))(\xi_i - \tilde{\xi}_i) \geq C\|\xi - \tilde{\xi}\|^p. \qquad (4.4)$$

Clearly, integrating (4.4) gives (4.1) with $\gamma = C$. Therefore (C3) ensures the uniform monotonicity of operator A.

The reader interested in this existence type result is referred to the paper by Lions [10] and the book by Ziedler [5]. Now we only give a practical conditions on the functions a_i that guaranteed condition (C3).

Proposition 4.1 *Suppose that $p \geq 2$ and a_i are continuously differentiable in variable ξ for all $i = 0, 1, 2, ...n$. Further assume that there exists a constant $\delta > 0$ such that for a.e $x \in \Omega$ and for all $\tilde{\xi}, \xi \in \mathbb{R}^{n+1}$ and $(z_0, z_1, z_2, ...z_n) \in \mathbb{R}^{n+1}$*

$$\sum_{j=0}^{n}\sum_{i=0}^{n} D_j a_i(x,\xi) z_i z_j \geq \delta \sum_{i=0}^{n} |\xi_i|^{p-2} z_i^2 \tag{4.5}$$

then (C3) holds.

To prove this proposition we first prove the following Lemma.

Lemma 4.2 *Let a, b be arbitrary in $\mathbb{R}$, and $s \geq 0$ then*

$$\int_0^1 |a + b\tau|^s d\tau \;\geq\; \frac{|b|^s}{2^s(s+1)} \tag{4.6}$$

Proof. For $b = 0$ the result follows trivially. Now assume that $b \neq 0$. Then the inequality (4.6) is equivalent to

$$\int_0^1 \left|\frac{a}{b} + \tau\right|^s d\tau \;\geq\; \frac{1}{2^s(s+1)}$$

and so without loss of generality we may suppose that $b = 1$. Let $a \in \mathbb{R}$ be fixed, then

$$\int_0^1 |a+\tau|^s d\tau = \int_a^{a+1} |t|^s \, dt \text{ where } \; t := a + \tau.$$

Case 1: For $0 \leq a \leq a + 1$, we have

$$\int_a^{a+1} |t|^s dt \;=\; \int_a^{a+1} t^s dt \;=\; \frac{(a+1)^{s+1} - a^{s+1}}{(s+1)}.$$

So we need only to show that $(a + 1)^{s+1} - a^{s+1} \geq 1$ for all $a \geq 0$ and $s \geq 0$. Now for fixed s consider the function $f : [0, +\infty) \longrightarrow \mathbb{R}$ defined by

$$f(a) = (a + 1)^{s+1} - a^{s+1} \text{ for all } a \in [0, +\infty).$$

Then we see that

$$f'(a) = (s + 1)((a + 1)^s - a^s) \geq 0 \text{ for all } a \in [0, +\infty),$$

which implies that f is increasing on $[0,+\infty)$, so that $f(a) \geq f(0) = 1$ for all $a \in [0,+\infty)$. Therefore we have

$$\frac{(a+1)^{s+1} - a^{s+1}}{s+1} \geq \frac{1}{s+1} \geq \frac{1}{2^s(s+1)}. \tag{4.7}$$

Case 2: For $a < a+1 \leq 0$, we have

$$\int_a^{a+1} |t|^s dt \;=\; \int_a^{a+1} (-t)^s dt \;=\; -\int_{-a}^{-(a+1)} y^s dy \;=\; \frac{(-a)^{s+1} - (-a-1)^{s+1}}{s+1}.$$

Let $k = -(a+1)$, then $-a = k+1$ and $0 \leq k < k+1$. The result follows from case 1.

Case 3: Otherwise, $a \leq 0 \leq a+1$. and then

$$\begin{aligned}\int_a^{a+1} |t|^s dt \;&=\; \int_a^0 (-t)^s dt \;+\; \int_0^{a+1} t^s dt \\ &=\; -\int_{-a}^0 y^s dy \;+\; \int_0^{a+1} t^s dt \\ &=\; \frac{(-a)^{s+1} + (a+1)^{s+1}}{s+1}.\end{aligned}$$

Studying the function $a \mapsto (-a)^{s+1} + (a+1)^{s+1}$ on $(-1,0)$ we see that the minimum is achieved at $a = -\frac{1}{2}$, thus we have

$$\frac{(-a)^{s+1} - (a+1)^{s+1}}{s+1} \geq \frac{1}{s+1} \geq \frac{1}{2^s(s+1)}.$$

Therefore for arbitrary $a \in \mathbb{R}$ we have

$$\int_0^1 |a+\tau|^s d\tau \geq \frac{1}{2^s(s+1)}. \tag{4.8}$$

□

We now give the proof of Proposition (4.1)

Proof of Proposition 4.1. For fixed $x \in \Omega$, $\tilde{\xi}, \xi \in \mathbb{R}^{n+1}$, define a function $f_i : [0,1] \longrightarrow \mathbb{R}$ by

$$f_i(\tau) = a_i(x, \tilde{\xi} + \tau(\xi - \tilde{\xi}), \; i = 0,1,2,...n.$$

Then by fundamental theorem of calculus, assumption (4.5) and Lemma (4.2) we have

$$\begin{aligned}
\sum_{i=0}^{n}(a_i(x,\xi)-a_i(x,\tilde{\xi}))(\xi_i-\tilde{\xi}_i) &= \sum_{i=0}^{n}(f_i(1)-f_i(0))(\xi_i-\tilde{\xi}_i) \\
&= \sum_{i=0}^{n}\int_0^1\sum_{j=0}^{n}D_j a_i(x,\tilde{\xi}+\tau(\xi-\tilde{\xi})(\xi_j-\tilde{\xi}_j)(\xi_i-\tilde{\xi}_i)d\tau \\
&= \delta\sum_{i=0}^{n}\int_0^1\|\tilde{\xi}+\tau(\xi-\tilde{\xi})\|^{p-2}(\xi_i-\tilde{\xi}_i)^2 d\tau \\
&= \delta\int_0^1\|\tilde{\xi}+\tau(\xi-\tilde{\xi})\|^{p-2}\sum_{i=0}^{n}(\xi_i-\tilde{\xi}_i)^2 d\tau \\
&= \delta\,\|\xi-\tilde{\xi}\|^2\int_0^1\|\tilde{\xi}+\tau(\xi-\tilde{\xi})\|^{p-2}d\tau \\
&\geq \delta\,\|\xi-\tilde{\xi}\|^2\int_0^1\Big|-\|\tilde{\xi}\|+\tau\|\xi-\tilde{\xi}\|\Big|^{p-2}d\tau \\
&\geq \frac{\delta}{2^{p-2}(p-1)}\|\xi-\tilde{\xi}\|^p.
\end{aligned}$$

Hence we have

$$\sum_{i=0}^{n}(a_i(x,\xi)-a_i(x,\xi))(\xi_i-\tilde{\xi}_i)\geq\frac{\delta}{2^{p-2}(p-1)}\|\xi-\tilde{\xi}\|^p. \tag{4.9}$$

Integrating (4.9) yields

$$\langle A(u_1)-A(u_2),u_1-u_2\rangle\geq\frac{\delta}{2^{p-2}(p-1)}\|u_1-u_2\|^p,$$

which gives condition (C3) with $C=\frac{\delta}{2^{p-2}(p-1)}$.

We now give some examples of uniformly monotone operators which fulfil also condition (C1) and (C2). In the sequel we always suppose $p\geq 2$

Example 1. Let $a_i(\xi)=\xi_i|\xi_i|^{p-2}$, $i=0,1,2,...n$. Then

$$\langle A(u),v\rangle=\int_\Omega\left(\sum_{i=1}^{n}D_iuD_iv|D_iu|^{p-2}+uv|u|^{p-2}\right)dx$$

For fixed $\xi\in\mathbb{R}^{n+1}$, the function $x\mapsto a_i(\xi)$ is constant in x. So it's measurable. Also the function a_i is continuous for each i. Thus for each i a_i is of Caratheodory type. We also have that

$$|a_i(\xi)| = |\xi_i||\xi_i|^{p-2} = |\xi_i|^{p-1} \leq \sum_{i=1}^{n}(|\xi_i|^2)^{\frac{p-1}{2}} = \|\xi\|^{p-1}.$$

Thus by taking $c = 2$ and $k \equiv 0$ we see that (C2) is satisfied.

Clearly, $D_i a_i(\xi) = (p-1)|\xi_i|^{p-2}$ and $D_j a_i(\xi) = 0$ for $i \neq j$. Hence we have

$$\sum_{j=0}^{n}\sum_{i=0}^{n} D_j a_i(\xi) z_i z_j = (p-1)\sum_{i=0}^{n} |\xi_i|^{p-2} z_i^2.$$

Therefore by proposition (4.1) we see that (C3) is satisfied.

Example 2. Now let

$$\begin{aligned} a_i(\xi) &= \xi_i\|\xi\|^{p-2} \quad (i = 1, 2, ...n), \\ a_0(\xi) &= \xi_0|\xi_0|^{p-2}. \end{aligned}$$

So we have

$$\langle A(u), v\rangle = \int_\Omega \left(\sum_{i=1}^{n} D_i u D_i v|Du|^{p-2} + uv|u|^{p-2}\right) dx.$$

In this case it can be easily seen that A is the weak form of the operator

$$u \ \mapsto \ -\Delta_p + u|u|^{p-2}.$$

Obviously (C1) is satisfied. Moreover

$$|a_i(\xi)| = |\xi_i|\|\xi\|^{p-2} \leq \underset{1\leq i\leq n}{max}\{|\xi_i|\}\|\xi\|^{p-2} \leq \alpha\|\xi\|\|\xi\|^{p-2} = \alpha\|\xi\|^{p-1},$$

where $\alpha > 0$. (Here we have used the fact that $\|\cdot\|_1$ and $\|\cdot\|_2$ in $\mathbb{R}^{n+1}$ are equivalent). So we have (C2) is satisfied. We also have that

$$\begin{cases} D_j a_i(\xi) = (p-2)\xi_j\xi_i\|\xi\|^{p-4}, & \text{for } i, j > 0, i \neq j \\ D_i a_i(\xi) = \|\xi\|^{p-2} + (p-2)\xi_i^2\|\xi\|^{p-4}, & \text{for } i > 0 \\ D_j a_0 = D_0 a_i(\xi) = 0, & \text{for } i > 0, j > 0 \\ D_0 a_0(\xi) = (p-1)\|\xi_0\|^{p-2}. & \end{cases}$$

Now

$$\begin{aligned}
\sum_{j=0}^{n}\sum_{i=0}^{n} D_j a_i(\xi) z_i z_j &= D_0 a_0(\xi) z_0^2 + \sum_{j=1}^{n}\sum_{i=1}^{n} D_j a_i(\xi) z_i z_j \\
&= (p-1)\|\xi_0\|^{p-2} z_0^2 + \sum_{i=1}^{n} \|\xi\|^{p-2} z_i^2 \\
&+ (p-2)\|\xi\|^{p-4} \sum_{j=1}^{n}\sum_{i=1}^{n} \xi_i \xi_j z_i z_j \\
&= (p-1)\|\xi_0\|^{p-2} z_0^2 + \sum_{i=1}^{n} \|\xi\|^{p-2} z_i^2 \\
&+ (p-2)\|\xi\|^{p-4} \left(\sum_{i=1}^{n} \xi_i z_i \right)^2 \\
&\geq (p-1)\|\xi_0\|^{p-2} z_0^2 + \sum_{i=1}^{n} \|\xi\|^{p-2} z_i^2 \\
&\geq (p-1)\|\xi_0\|^{p-2} z_0^2 + \sum_{i=1}^{n} |\xi_i|^{p-2} z_i^2 = \sum_{i=0}^{n} |\xi_i|^{p-2} z_i^2 .
\end{aligned}$$

Thus, from proposition (4.1) it follows that the operator A is uniformly monotone.

Bibliography

[1] V. Barbu, *Analysis and control of Nonlinear Infinite Dimensionl Systems,* School of Mathematics, University of Iasi, Iasi Romania, November 1991.

[2] V. Barbu, *Nonlinear Differential Equations of Monotone types in Banach Spaces,* School of Mathematics, University of Iasi, Iasi Romania, September 2009.

[3] C.E Chidume, *Geometric Properties of Banach Spaces and Nonlinear Iterations,* International Center for Theoretical Physics, Triesty, Italy, June 2008.

[4] C.E Chidume, *Applicable Functional Analysis,* International Center for Theoretical Physics, Triesty, Italy, July 2006.

[5] E. Ziedler, *Nonlinear functional analysis and its applications, Part II: Monotone operators* (springer-Verlag, Berlin/New York, 1985).

[6] K. Ito and F. Kappel, *Evolution Equations and Approximations,* North California University, USA, December 2001.

[7] S. Simons and C. Zalinescu, *A New Proof for Rockerfellar's Characterizations of Maximal Monotone Operators, Proc. Amer. math. Soc.* 131(2004), pp. 2969-2972.

[8] H. Brezis, *Monotone Operators, Nonlinear Semigroups and applications, Proc.. of the Int. Cong. of Math.* Vancouver, (1974), 249-255.

[9] A. Besenyei, *On Uniformly Monotone Operators arising in Nonlinear Elliptic and Parabolic Problems,* Annales Univ. Sci. Budapest. Etovos Sect. Math. (2011).

[10] J. Leray and Jacques-L. Lions, *Quelques results de Visik sur les problemes elliptiques non lineaires par les methodes de Minty Browder Bulletin de la S.M.F,* tome 93 (1965), p.97-107.

[11] N. Djitte, *Lecture Notes on Sobolev Spaces and Linear Elliptic Partial Differential Equations*, African University of Science and Technology, Abuja, 2011.

[12] N. Djitte, *Lecture Notes on Variational Methods*, African University of Science and Technology, Abuja, 2011.

[13] Cf. Dinca Jebelean -Mahawin, *Variational and Topological Methods for Drichilet Problem with P-Laplacian,* Portugalia Mathematica. Vol. 58 Fasc.3 - 2001. Nova series 339-378.