EXISTENCE AND UNIQUENESS OF SOLUTIONS OF INTEGRAL EQUATIONS OF HAMMERSTEIN TYPE

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Dedication

This thesis is dedicated to God, the author and the finisher of the faith and the sustainer of the souls.

Preface

This Project centres on integral equations of Hammerstein type, abstract Hammerstein equations and monotone operators in Banach spaces. Let X be a real Banach space, X^* its dual, A a linear map of X into X^* and N a nonlinear map of X^* into X. We study the abstract Hammerstein equation,

$$w + ANw = 0, \quad w \in X^*,$$

and theorems that establish general results on the existence and uniqueness of solutions of the Hammerstein equations.

Hammerstein equation covers a large variety of areas and is of much interest to a wide audience due to the fact that it has applications in numerous areas. Several problems that arise in differential equations (ordinary and partial), for instance, elliptic boundary value problems whose linear parts possess Green's function can be transformed into the Hammerstein integral equations. Equations of the Hammerstein type play a crucial role in the theory of optimal control systems and in automation and network theory (see e.g., Dolezale [12]).

Problem of existence of solutions arises naturally in different areas of life. There are methods that help one to ascertain if there exists a solution to a particular problem. In general fixed point theorems, Banach Contraction Mapping Principle and Schauder-Tychonov Fixed Point Theorem are being used. However none of the these theorems is applicable here because our operator is not compact or contractive.

The concept of monotone operators, introduced in the 1960s, has proved to be very effective in obtaining existence results in nonlinear problems. One of the reasons is certainly lack of compactness among the basic requirements. Also, compactness is not always easy to check and it does represent a rather severe restriction on the operator. Many researchers have successfully applied monotonicity concepts to the Hammerstein equations.

We study factorization of operators and variational methods as they apply to solvability of the Hammerstein equation. Chapter one is the general introduction. We use factorization of operators in chapter two and variational methods in chapter three to establish the general results on the existence and uniqueness of solutions of Hammerstein equations.

Assuming suitable growth conditions on N, existence results were obtained under the following conditions on X, A and N. In chapter two: X is a Banach space, A is monotone, angle-bounded, continuous and linear, N is hemicontinuous. In chapter three: X is a Banach space, A is linear, monotone and symmetric, N is a potential.

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ABSTRACT

Let X be a real Banach space, X^* its conjugate dual space. Let A be a monotone angle-bounded continuous linear mapping of X into X^* with constant of angle-boundedness $c \geq 0$. Let N be a hemicontinuous (possibly non-linear) mapping of X^* into X such that for a given constant $k \geq 0$,

$$\langle v_1 - v_2,\ Nv_1 - Nv_2 \rangle \geq -k\|v_1 - v_2\|_{X^*}^2$$

for all v_1 and v_2 in X^*. Suppose finally that there exists a constant R with $k(1 + c^2)R < 1$ such that for $u \in X$

$$\langle Au, u \rangle \leq R\|u\|_X^2.$$

Then, there exists exactly one solution w in X^* of the nonlinear equation

$$w + ANw = 0.$$

Existence and uniqueness is also proved using variational methods.

Contents

CHAPTER 1

General Introduction

1.1 Introduction

The contribution of this thesis falls within the general area of nonlinear functional analysis. Within this area, our attention is focused on the topic: "Existence and Uniqueness of Solutions of Nonlinear Hammerstein Integral Equations" in Banach spaces. We study theorems that establish existence and uniqueness of solutions of these equations using factorization of operators and variational methods.

Several classical problems in the theory of differential equations lead to integral equations. In many cases, these equations can be dealt with in a more satisfactory manner using the integral form than directly with differential equations.

Interest in Hammerstein equations stem mainly from the fact that several problems that arise in differential equations, for instance, elliptic boundary value problems whose linear parts possess Green's function can, as a rule be transformed into a nonlinear integral equation of Hammerstein type. Elliptic boundary value problems are a class of problems which do not involve time variable but only depend on the space variables. That is, they are class of problems which are typically associated with steady state behaviour. An example is a Laplace's equation:

$$\nabla^2 u = 0 \text{ e.g } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ in } 2D \ .$$

Consequently, solvability of such differential equations is equivalent to the solvability of the corresponding Hammerstein equation.

1.2 Definition and examples of some basic terms

In this section, definitions of basic terms used are given.

Throughout this chapter, X denotes a real Banach space and X^* denotes its corresponding dual. We shall denote by the pairing $\langle x^*, x\rangle$ or $x^*(x)$ the value of the functional $x^* \in X^*$ at $x \in X$. The norm in X is denoted by $\|.\|$, while the norm in X^* is denoted by $\|.\|_*$. If there is no danger of confusion, we omit the asterisk and denote both norms in X and X^* by the symbol $\|.\|$. We shall use the symbol $\rightarrow$ to indicate strong and $\rightharpoonup$ to indicate weak convergence. We shall also use $\overset{w^*}{\rightarrow}$ to indicate the weak-star convergence.

The first term we define is monotone map. The concept of monotonicity pertains to nonlinear functional analysis, and its use in the theory of functional equations (ordinary differential equations, integral equations, integrodifferential equations, delay equations) is probably the most powerful method in obtaining existence theorems.

Definition 1.2.1 (Monotone Operator): *A map $A : D(A) \subset X \rightarrow 2^{X^*}$ is said to be monotone if $\forall\ x, y \in D(A),\ x^* \in Ax,\ y^* \in Ay$, we have*

$$\langle x^* - y^*,\ x - y\rangle \geq 0.$$

From the definition above, a single-valued map $A : D(A) \subset X \rightarrow X^*$ is monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0,\ \forall\ x, y \in D(A).$$

Remark 1.2.1 *For a linear map A, the above definition reduces to*

$$\langle Au, u\rangle \geq 0\ \forall\ u \in D(A).$$

The following are some examples of monotone operators.

Example 1.2.1 *Every nondecreasing function on $\mathbb{R}$ is monotone.*

Proof.
Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a nondecreasing function. Then for arbitrary $x, y \in \mathbb{R}$, both $(f(x) - f(y))$ and $(x - y)$ have the same sign. Thus we see that $\langle f(x) - f(y), x - y\rangle = (f(x) - f(y))(x - y) \geq 0\ \forall\ x, y\ \in \mathbb{R}$. Hence, f is monotone.

Example 1.2.2 *Let $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined as $h(x, y) = (2x, 5y)$, $\forall\ (x, y) \in \mathbb{R}^2$. Then h is montone.*

Proof.
For arbitrary $(x_1, y_1),\ (x_2,\ y_2)\ \in \mathbb{R}^2$, we have

$$\langle h(x_1, y_1) - h(x_2,\ y_2), (x_1,\ y_1) - (x_2,\ y_2)\rangle \ = \ 2(x_1 - x_2)^2 + 5(y_1 - y_2)^2 \geq 0.$$

Thus, h is monotone.

Example 1.2.3 *Let H be a real Hilbert space, I is the identity map of H and $T : H \to H$ be a non-expansive map (i.e $\|Tx - Ty\| \leq \|x - y\| \ \forall \ x, y \in H$). Then the operator $I - T$ is monotone.*

Proof.
Let $x, y \ \in H$, then

$$\begin{aligned}
\langle (I-T)x - (I-T)y, \ x-y\rangle &= \langle (x-y) - (Tx - Ty), \ x - y\rangle \\
&= \|x-y\|^2 - \langle Tx - Ty, \ x-y\rangle \\
&\geq \|x-y\|^2 - \|Tx-Ty\|.\|x-y\| \\
&\geq \|x-y\|^2 - \|x-y\|^2 = 0 \text{ (T is nonexpansive).}
\end{aligned}$$

Thus we have that $I - T$ is monotone on H.

Example 1.2.4 *Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\bar{x} = \begin{pmatrix} x \\ y \end{pmatrix}$. Consider the function $g : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $g(\bar{x}) = A\bar{x}$. Then g is monotone.*

Proof.
Since g is linear, by remark (1.2.1) it suffices to show that $\langle g(\bar{x}), \bar{x}\rangle \geq 0$. For arbitrary $\bar{x} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$, we have $A\bar{x} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$.

Thus $\langle g(\bar{x}), \bar{x}\rangle = \langle A\bar{x}, \bar{x}\rangle = x^2 + 0 = x^2 \geq 0$. Hence g is monotone.

Example 1.2.5 *Let X be a real Banach space. The duality map $J : X \to 2^{X^*}$ defined by*

$$Jx := \{x^* \in X^* : \langle x, x^*\rangle = \|x\|.\|x^*\|, \quad \|x\| = \|x^*\|, \quad x \in X\}$$

is monotone.

Proof.
Let $x, y \ \in X$ and $x^* \in Jx, y^* \in Jy$. Then

$$\begin{aligned}
\langle x^* - y^*, x - y\rangle &= \langle x^* - y^*, x\rangle - \langle x^* - y^*, y\rangle \\
&= \langle x^*, x\rangle - \langle y^*, x\rangle - \langle x^*, y\rangle + \langle y^*, y\rangle \\
&= \|x\|^2 + \|y\|^2 - \langle y^*, x\rangle - \langle x^*, y\rangle \\
&\geq \|x\|^2 + \|y\|^2 - \|y^*\|.\|x\| - \|x^*\|.\|y\| \\
&= \|x\|^2 + \|y\|^2 - 2\|x\|.\|y\| \\
&= (\|x\| - \|y\|)^2 \geq 0.
\end{aligned}$$

Thus, J is monotone.

Example 1.2.6 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be convex and proper. The subdifferential of f, $\partial f : X \to 2^{X^*}$ defined as*

$$\partial f(x) = \begin{cases} \{x^* \in X^* : \langle y - x, x^*\rangle \leq f(y) - f(x), \ y \in X\}, & \text{if } f(x) \neq \infty \\ \emptyset, & \text{if } f(x) = \infty, \end{cases}$$

is monotone.

Proof.
Let $x, y \in X,\ x^* \in \partial f(x)$ and $y^* \in \partial f(y)$.

$$x^* \in \partial f(x) \Rightarrow \langle y - x,\ x^* \rangle \leq f(y) - f(x)\ \ \forall\ \ y \in X. \tag{1.2.1}$$

$$\begin{aligned} y^* \in \partial f(y) \quad &\Rightarrow \quad \langle x - y,\ y^* \rangle \leq f(x) - f(y)\ \ \forall\ \ x \in X \\ &\Rightarrow \quad -\langle y - x,\ y^* \rangle \leq f(x) - f(y)\ \ \forall\ \ x \in X. \end{aligned} \tag{1.2.2}$$

Adding inequalities (1.2.1) and (1.2.2), we have

$$\langle y - x,\ x^* \rangle - \langle y - x,\ y^* \rangle \leq 0.$$

This implies that $\langle y - x, x^* - y^* \rangle \leq 0$, i.e $\langle x - y, x^* - y^* \rangle \geq 0$.

Definition 1.2.2 (Hemicontinuity): *A mapping $A : D(A) \subset X^* \to X$ is said to be hemicontinuous if it is continuous from each line segment of X^* to the weak topology of X. That is, $\forall\ u \in D(A)$, $\forall\ v \in X^*$ and $(t_n)_{n\geq 1} \subset \mathbb{R}^+$ such that $t_n \to 0^+$ and $u + t_n v \in D(A)$ for n sufficiently large, we have $A(u + t_n v) \rightharpoonup A(u)$.*

Proposition 1.2.1 *Let X denote a Banach space and X^* its corresponding dual. Let $A : D(A) \subset X^* \to X$ be a continuous mapping . Then A is hemicontinuous.*

Proof
Let $u \in D(A)$, $v \in X^*$, $(t_n)_{n\geq 1}$ be a sequence of positive numbers such that $t_n \to 0^+$ as $n \to \infty$ and $(u + t_n v) \in D$ for n large enough. We observe that $(u + t_n v) \to u$ as $n \to \infty$ because $t_n \to 0^+$ as $n \to \infty$. By the continuity of A, we have $A(u + t_n v) \to A(u)$ as $n \to \infty$. Since strong convergence implies weak convergence we have $A(u + t_n v) \rightharpoonup A(u)$ as $n \to \infty$. Hence A is hemicontinuous.

Remark 1.2.2 *The converse of proposition (1.2.1) is false.*

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(x, y) \;=\; \begin{cases} (\frac{x^2 + xy^2}{x^2 + y^4}, x), & \text{if } (x, y) \neq (0, 0) \\ (1, 0), & \text{if } (x, y) = (0, 0). \end{cases}$$

Clearly, f is not continuous at $(0, 0)$. For,

$f(x, 0) = (\frac{x^2}{x^2}, x) = (1, x)$ for all $x \neq 0$. This implies $\lim\limits_{x\to 0} f(x, 0) = (1, 0)$.

$f(0, y) = (0, 0)$, $\forall y \neq 0$. This implies $\lim\limits_{y\to 0} f(0, y) = (0, 0)$. Thus, the limit does not exist at $(0, 0)$. Hence, f is not continuous at (0,0).

However, f is hemicontinuous. Indeed, let $u = (0,0)$, $v = (v_1, v_2)$ and $\{t_n\}_{n\geq 1}$ be arbitrary such that $t_n \to 0^+$ as $n \to \infty$. Then,

$f(u + t_n v) = f(t_n v_1, t_n v_2)) = \left(\frac{v_1^2 + t_n v_1 v_2^2}{v_1^2 + t_n^2 v_2^4}, t_n v_1\right) \to (1,0)$, as $n \to \infty$. Therefore, $\lim\limits_{n\to\infty} f(u + t_n v) = (1,0) = f(0,0)$. Thus, $f(u + t_n v) \to f(u)$ as $t_n \to 0^+$. Hence, f is hemicontinuous on $\mathbb{R}^2$ since strong and weak convergence are the same on $\mathbb{R}^2$.

Definition 1.2.3 (Coercivity): *An operator $A : X \to X^*$ is said to be coercive if for any $x \in X$, $\frac{\langle x, Ax\rangle}{\|x\|} \to \infty$ as $\|x\| \to \infty$.*

Example 1.2.7 *Let H be a real Hilbert space and $f : H \to H$ be defined by $f(x) = \frac{1}{2}u$. Then, f is coercive.*

Proof.
Let $x \in H$ be arbitrary. Then,

$$\frac{\langle f(x), x\rangle}{\|x\|} = \frac{\frac{1}{2}\langle x, x\rangle}{\|x\|} = \frac{\frac{1}{2}\|x\|^2}{\|x\|} = \frac{1}{2}\|x\| \to +\infty \text{ as } \|x\| \to \infty.$$

Hence f is coercive.

Definition 1.2.4 (Symmetry): *Let $A : X \to X^*$ be a bounded linear mapping. A is said be symmetric if for all u and v in X, we have $\langle Au, v\rangle = \langle Av, u\rangle$.*

Example 1.2.8 *Let $A : l_2(\mathbb{R}) \to l_2(\mathbb{R})$ be a map defined by $Au = \frac{1}{2}u$. Then A is symmetric.*

Proof.
For arbitrary $u, v \in l_2$,

$$\begin{aligned}
\langle Au, v\rangle &= \left\langle \frac{1}{2}u, v\right\rangle \\
&= \frac{1}{2}\langle (u_1, u_2, ...), (v_1, v_2, ...)\rangle = \frac{1}{2}\sum_{i=1}^{\infty} u_i v_i \\
&= \frac{1}{2}\sum_{i=1}^{\infty} v_i u_i = \frac{1}{2}\langle (v_1, v_2, ...), (u_1, u_2, ...)\rangle \\
&= \left\langle \frac{1}{2}v, u\right\rangle = \langle u, Av\rangle .
\end{aligned}$$

Hence A is symmetric.

Definition 1.2.5 (Skew-symmetry): *Let $A : X \to X^*$ be a bounded linear mapping. A is said be skew-symmetric if for all u and v in X, we have $\langle Au, v\rangle = -\langle Av, u\rangle$.*

Definition 1.2.6 (Angle-boundedness): *Let $A : X \to X^*$ be a bounded monotone linear mapping . A is said be angle-bounded with constant $c \geq 0$ if for all u, v in X, $|\langle Au, v\rangle - \langle Av, u\rangle| \leq 2c\{\langle Au, u\rangle\}^{\frac{1}{2}}\{\langle Av, v\rangle\}^{\frac{1}{2}}$. (This is well defined since $\langle Au, u\rangle \geq 0$ and $\langle Av, v\rangle \geq 0$ by the linearity and monotonicity of A).*

Example 1.2.9 *A symmetric map. It follows that every symmetric mapping A of X into X^* is angle-bounded with constant of angle-boundedness $c = 0$.*

Definition 1.2.7 (Adjoint Operators): *Let X and Y be normed linear spaces and $A \in B(X,\ Y)$. The adjoint of A, denoted by A^*, is the operator $A^* : Y^* \to X^*$ defined by $\langle A^*y^*, x\rangle = \langle y^*, Ax\rangle$ for all $y^* \in Y^*$ and all $x \in X$.*

We note that A^* is well-defined. Indeed, $\forall\ y^* \in Y^*,\ x_1, x_2 \in X$ and $\alpha \in \mathbb{R}$, we have

$$\begin{aligned}\langle A^*y^*, \alpha x_1 + x_2\rangle &= \langle y^*, A(\alpha x_1 + x_2)\rangle = \langle y^*, \alpha Ax_1\rangle + \langle y^*, Ax_2\rangle \\ &= \alpha\langle y^*, Ax_1\rangle + \langle y^*, Ax_1\rangle\end{aligned}$$

which shows that A^*y^* is linear.

For boundedness, given $y^* \in Y^*$ and $x \in X$,

$$\begin{aligned}|\langle A^*y^*, x\rangle| &= |\langle y^*, Ax\rangle| \\ &\leq \|y^*\|.\|Ax\| \text{ since } y^* \in Y^*. \\ &\leq \|y^*\|.\|A\|.\|x\| \text{ since } A \in B(X, Y).\end{aligned}$$

Therefore, for all $y^* \in Y^*$,

$$|\langle A^*y^*, x\rangle| \leq K_{y^*}\|x\|\ \forall\ x \in X, \text{ where } K_{y^*} = \|y^*\|.\|A\| \geq 0.$$

Hence, for all $y^* \in Y^*, A^*y^* \in X^*$.

Theorem 1.2.1 *Let $A : X \to Y$ be a bounded linear maps with adjoint A^*. Then,*

(a) $A^* \in B(Y^*, X^*)$;

(b) $\|A\| = \|A^*\|$.

Proof.

(a) Let $y^*, z^* \in Y^*$ and $\alpha \in \mathbb{R}$. We show that

$$A^*(\alpha y^* + z^*) = \alpha A^*y^* + A^*z^*,$$

i.e
$$\forall\ x \in X,\ \ \langle A^* (\alpha y^* + z^*), x\rangle = \alpha \langle A^* y^*, x\rangle + \langle A^* z^*, x\rangle .$$

Let $x \in X$. Then

$$\begin{aligned}\langle A^* (\alpha y^* + z^*), x\rangle &= \langle \alpha y^* + z^*, Ax\rangle = \alpha \langle y^*, Ax\rangle + \langle z^*, Ax\rangle \\ &= \alpha \langle A^* y^*, x\rangle + \langle A^* z^*, x\rangle .\end{aligned}$$

So, A^* is linear.

Furthermore, for any $y^* \in Y^*$ and $x \in X$,

$$|\langle A^* y^*, x\rangle| = |\langle y^*, Ax\rangle| \leq \|y^*\|.\|A\|.\|x\|, \text{ since } A \in B(X, Y)\ .$$

Thus, $\|A^* y^*\| = \sup\limits_{\|x\|=1} |\langle A^* y^*,\ x\rangle| \leq \|A\|.\|y^*\|$. Therefore, $\|A^* y^*\| \leq K\|y^*\|$, where $K = \|A\| \geq 0$. Hence $A^* \in B(Y^*, X^*)$.

(b)

$$\begin{aligned}\|A\| &= \sup_{\|x\|=1} \|Ax\| = \sup_{\|x\|=1} \left(\sup_{\|y^*\|=1} \langle y^*, Ax\rangle \right) \\ &= \sup_{\|x\|=1} \left(\sup_{\|y^*\|=1} \langle A^* y^*, x\rangle \right) \\ &= \sup_{\|y^*\|=1} \left(\sup_{\|x\|=1} \langle A^* y^*, x\rangle \right) \\ &= \sup_{\|y^*\|=1} \|A^* y^*\| = \|A^*\|.\end{aligned}$$

Definition 1.2.8 (Weak Topology): *Let (X, ω) denote a Banach space endowed with the weak topology. For an arbitrary sequence $\{x_n\}_{n\geq 1} \subset X$ and $x \in X$, we say that $\{x_n\}$ converges weakly to x if $f(x_n) \to f(x)$ for each $f \in X^*$. We denote this by $x_n \rightharpoonup x$.*

Definition 1.2.9 (Weak Star Topology): *Let (X^*, ω^*) denote a Banach space endowed with the weak star topology. For an arbitrary sequence $\{f_n\}_{n\geq 1} \subset X^*$ and $f \in X^*$ we say that $\{f_n\}$ converges to f in weak-star topology, denoted $f_n \xrightarrow{\omega^*} f$, if $f_n(x) \to f(x)$ for each $x \in X$.*

Proposition 1.2.2 *Let $\{x_n\}$ be a sequence and x a point in X. Then the following hold.*

(a) $x_n \to x \quad \Rightarrow \quad x_n \rightharpoonup x$;

(b) $x_n \rightharpoonup x \quad \Rightarrow \{x_n\}$ *is bounded and* $\|x\| \leq \liminf \|x_n\|$;

(c) $x_n \rightharpoonup x$ (in X), $f_n \to f$ (in X^*) $\Rightarrow$ $f_n(x_n) \to f(x)$ (in $\mathbb{R}$).

Definition 1.2.10 (Reflexive Space): *Let X be a Banach space and let $J : X \to X^{**}$ be the canonical injection from X into X^{**}, that is $\langle J(x), f\rangle = \langle f, x\rangle$, $\forall\ x \in X, f \in X^*$. Then X is said to be* **reflexive** *if J is surjective, i.e $J(X) = X^{**}$.*

Definition 1.2.11 (Uniformly convex Banach spaces): *A Banach space X is called uniformly convex if for any $\epsilon \in (0, 2]$, there exists a $\delta = \delta(\epsilon) > 0$ such that if $x, y \in X$, with $\|x\| \leq 1, \|y\| \leq 1$ and $\|x - y\| \geq \epsilon$, then $\|\frac{1}{2}(x + y)\| \leq 1 - \delta$.*

Hilbert spaces, L_p and l_p spaces, $1 < p < \infty$ are examples of uniformly convex spaces.

Definition 1.2.12 (Strictly convex spaces): *A normed linear space X is said to be strictly convex if for all $x,\ y \in X$, $x \neq y$, $\|x\| = \|y\| = 1$, we have $\|\alpha x + (1 - \alpha)y\| < 1$ for all $\alpha \in (0, 1)$.*

Theorem 1.2.2 Milman-Pettis Theorem: *Every uniformly convex Banach space X is reflexive.*

For the proof of theorem (1.2.2), see, for instance, Chidume [1].

Definition 1.2.13 ($\sigma-$algebra): *A collection $\mathcal{M}$ of subsets of a nonempty set Ω is called a $\sigma-$algebra if*

(a) $\phi,\ \Omega \in \mathcal{M}$,

(b) $A \in \mathcal{M} \to A^c \in \mathcal{M}$,

(c) $\cup_{n=1}^{\infty} A_n \in \mathcal{M}$ whenever $A_n \in \mathcal{M}\ \forall\ n$.

Definition 1.2.14 (Measurable Space): *If $\mathcal{M}$ is a $\sigma-$algebra of Ω, then the pair $(\Omega,\ \mathcal{M})$ is referred to as a measurable space.*

Definition 1.2.15 (Measure): *A measure on $(\Omega,\ \mathcal{M})$ is a function $\mu\ :\ \mathcal{M} \to [0,\ \infty]$ such that*

(a) $\mu(A) \geq 0$ for all $A \in \mathcal{M}$;

(b) $\mu(\phi) = 0$;

(c) if $A_i \in \mathcal{M}$ are pairwise disjoint, then $\mu\left(\cup_i^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$.

Definition 1.2.16 (Measure Space): *If $\mathcal{M}$ is a $\sigma-$algebra of subsets of Ω, and μ is a measure on $\mathcal{M}$, then the tripple $(\Omega,\ \mathcal{M},\ \mu)$ is referred to as a measure space.*

Definition 1.2.17 (Measurable Functions): *Let* $(\Omega,\ \mathcal{M})$ *be a measurable space. A function* $f : \Omega \to \mathbb{R}$ *is measurable or* $\mathcal{M}-$*measurable if the set* $\{x \in \Omega : f(x) > \alpha\} \in \mathcal{M}$ *for all* $\alpha \in \mathbb{R}$.

Definition 1.2.18 ($\sigma-$finite) : *A measure space* $(\Omega,\ \mathcal{M},\ \mu)$ *is said to be* $\sigma-$*finite if there exists a countable family* $(\Omega_n)_{n\geq 1}$ *in* $\mathcal{M}$ *such that* $\Omega = \cup_{n=1}^{\infty}\Omega_n$ *and* $\mu(\Omega_n) < \infty, \ \ \forall \ \ n$.

Definition 1.2.19 (Green's Function): *This is a function associated with a given boundary value problem, which appears as an integrand for an integral representation of the solution of the problem.*

Let $\mathcal{L}$ be a differential operator and assume that

$$\mathcal{L}(y) = \sum_{p=0}^{n} a_P(t)y^{(p)}(t) = a_n(t)y^n(t) + ... + a_(t)y^{(1)}(t) + a_0(t)y(t).$$

Suppose that $a_n(t)$ is not zero on $[0, 1]$ and that each term of the sequence $a_p(t), p = 0, ..., n$, has at least n continuous derivatives. Also suppose that B is the given boundary conditions associated with $\mathcal{L}$ and denote by M, the manifold associated with $(\mathcal{L}, B)$. (Manifold simply refers to the differential equation together with the associated boundary conditions.) We present the algorithm for constructing the Green's function, $G(t, x)$ for nth order equations. For $x \in [0, 1]$, we denote by x^-, the values of $t \in [0, x)$ and by x^+, the values of $t \in (x, 1]$.

(a) $\mathcal{L}\left(G(., x)\right)(t) = 0$ for $0 < t < x$ and for $x < t < 1$;

(b) $G(., x)$ is in M;

(c) for $0 \leq p \leq n-2$, $\frac{\partial^p G(t,x)}{\partial t^p}/_{t=x^+} = \frac{\partial^p G(t,x)}{\partial t^p}/_{t=x^-}$;

(d) $\frac{\partial^{n-1} G(t,x)}{\partial t^{n-1}}/_{t=x^+} - \frac{\partial^{n-1} G(t,x)}{\partial t^{n-1}}/_{t=x^-} = \frac{1}{a_n(x)}$.

Definition 1.2.20 (Carathéodory Condition): *Let* m *and* n *be positive integers,* Ω *be a nonempty subset of* $\mathbb{R}^m$ *and let* f *be a function from* $\Omega \times \mathbb{R}^n$ *into* $\mathbb{R}$. *A function* $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ *is said to satisfy the Carathéodory conditions if*

(i) $f(x,\ .) :\ \mathbb{R}^n \to \mathbb{R}$ *is a continuous function for almost all* $x \in \Omega$;

(ii) $f(.,\ u) :\ \Omega \to \mathbb{R}$ *is a measurable function for all* $u \in \mathbb{R}^n$.

Definition 1.2.21 (Nemystkii Operators): *Let* f *be a function from* $\Omega \times \mathbb{R}^n$ *into* $\mathbb{R}$. *We denote by* $\mathcal{F}(X, Y)$, *the set of all maps from* X *to* Y. *The Nemystkii operator associated to* f *is the operator* $N_f : \mathcal{F}(\Omega, \mathbb{R}^n) \to \mathcal{F}(\Omega, \mathbb{R})$ *defined by*

$$u \mapsto N_f(u)$$

where $(N_f u)(x) = f\left(x, u(x)\right)\ \forall\ u \in \mathcal{F}\left(\Omega,\ \mathbb{R}^n\right),\ \forall\ x \in \Omega$. *For simplicity, we shall write* $Nu_f(x)$ *instead of* $(N_f u)(x)$.

Example 1.2.10 *Given a map* $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *defined by*

$$f(x, s) = |s| \ \forall \ (x, s) \in \mathbb{R} \times \mathbb{R},$$

the Nemystkii operator associated to f *is given by the expression* $N_f u(x) = |u(x)|$ *for any map* $u : \mathbb{R} \to \mathbb{R}$ *and for any* $x \in \mathbb{R}$.

Example 1.2.11 *Given a map* $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *defined by*

$$g(x, s) = xe^s \ \forall \ (x, s) \in \mathbb{R} \times \mathbb{R},$$

the Nemystkii operator associated to g *is given by the expression* $N_f u(x) = xe^{u(x)}$ *for any map* $u : \mathbb{R} \to \mathbb{R}$ *and for any* $x \in \mathbb{R}$.

Observe that by the continuity of f and g, N_f and N_g map the set of real-valued continuous function on Ω; $C(\Omega)$ into itself. Moreover, they map the set of real-valued measurable function into itself.

1.3 Hammerstein Equations

A nonlinear integral equation of Hammerstein type on Ω is one of the form

$$u(x) + \int_\Omega k(x, y) f(y, u(y)) dy = h(x) \tag{1.3.1}$$

where dy stands for a σ-finite measure on the measure space Ω; the kernel k is defined on $\Omega \times \Omega$, f is a real-valued function defined on $\Omega \times \mathbb{R}$ and is in general nonlinear and h is a given function on Ω. If we define the operator A by

$$Av(x) = \int_\Omega k(x, y) v(y) dy, \tag{1.3.2}$$

and N_f to be the Nemystkii operator associated with f:

$$N_f u(x) = f(x, u(x)), \tag{1.3.3}$$

then the integral equation (1.3.1) can be put in functional equation form as follows:

$$u + AN_f u = 0, \tag{1.3.4}$$

where without loss of generality, we have taken $h \equiv 0$.

For $h \neq 0$, we have

$$\begin{aligned} u + AN_f u &= h \\ &\Rightarrow u - h + AN_f u = 0 \\ &\Rightarrow w + AN_f(w + h) = 0 \ \text{ where } \ w = u - h. \end{aligned}$$

Thus

$$w + A\bar{N}_f w = 0 \ \text{ where } \ \bar{N}_f w = N_f(w + h).$$

We consider, as an example of these, the forced oscillations of finite amplitude of a pendulum (see, e.g, Pascali and Sburlan [2], Chapter IV)

Example 1.3.1 *Consider an inhomogeneous differential equation given by*

$$\begin{cases} \frac{d^2v}{dt^2} + a^2 sin\ v(t) = z(t), & t \in [0,1] \\ v(0) = v(1) = 0. \end{cases} \tag{1.3.5}$$

The amplitude of oscillation $v(t)$ is a solution of the problem, where the driving force z(t) is periodic and odd. The constant $a \neq 0$ depends on the length of the pendulum and on gravity.

We begin the computation by computing the Green's function for the 2nd order equation,

$$v''(t) = 0, \quad v(0) = v(1) = 0. \tag{1.3.6}$$

According to the definition (1.2.19), since $n = 2$, the algorithm for computing $G(t,x)$ is given as:

(a) $\mathcal{L}\left(G(.,x)\right)(t) = 0$ *for* $0 < t < x$ *and for* $x < t < 1$*;*

(b) $G(.,x)$ *is in* M*;*

(c) $G(.,x)$ *is a continuous function;*

(d) $\frac{\partial G(t,x)}{\partial t/_{t=x^+}} - \frac{\partial G(t,x)}{\partial t}/_{t=x^-} = \frac{1}{a_2(x)}$.

The general solution of the homogeneous equation $v'' = 0$ is given by

$$v(t) = a_1 + a_2 t$$

where a_1 and a_2 are constants.
Thus, following the step(a), we seek the Green's function in the form

$$G(t,x) = \begin{cases} A + Bt, & 0 \leq t \leq x \\ C + Dt, & x \leq t \leq 1 \end{cases} \tag{1.3.7}$$

where A,B,C and D are functions of the parameter x.

Step(b) requires that $G(.,x)$ be in M. Therefore, we evaluate $G(0,x) = 0$ and $G(1,x) = 0$. The implications of this are that

$$A = 0 \text{ and } C + D = 0. \tag{1.3.8}$$

Step(c) requires that $G(.,x)$ be a continuous function, that is
$G(x^+,x) = G(x^-,x) \Rightarrow (C - A) + (D - B)x = 0.$
Since $A = 0$ (from (1.3.8)), we have

$$C + (D - B)x = 0 \tag{1.3.9}$$

Step(d) requires that $G_t/_{t=x^+} - G_t/_{t=x^-} = 1$ (since $a_2(x) = 1$). Thus

$$D - B = 1 \tag{1.3.10}$$

Solving ((1.3.8), (1.3.9) and (1.3.10)) for the three three unknowns B, C and D, we have that $B = x - 1,\ C = -x$ and $D = x$.
By substituting for the values of A, B, C and D into (1.3.7), we obtain

$$G(t,x) = \begin{cases} t(1-x), & 0 \le t \le x \\ -x + xt, & x \le t \le 1 \end{cases} \tag{1.3.11}$$

Equivalently, the Green's function for the given boundary value problem is the triangular function given by :

$$G(t,x) = \begin{cases} t(1-x), & 0 \le t \le x \\ x(1-t), & x \le t \le 1 \end{cases} \tag{1.3.12}$$

Equation (1.3.5) is equivalent to the nonlinear integral equation

$$v(t) = -\int_0^1 G(t,x)\left[z(x) - a^2 sin\ v(x)\right] dx \tag{1.3.13}$$

If $\int_0^1 G(t,x)z(x)dx = g(t)$ and $v(t) + g(t) = u(t)$, then equation (1.3.13) can be written as the integral equation

$$u(t) + \int_0^1 G(t,x) f(x, u(x))dx = 0. \tag{1.3.14}$$

where $f(x, u(x)) = a^2 sin\,[u(x) - g(x)]$.

Equation (1.3.14) is a homogeneous integral equation of Hammerstein type.

CHAPTER 2

Existence and Uniqueness Results Using Factorization of Operators

Let N denote the Nemystkii operator associated with f. We recall that the Hammerstein integral equation (1.3.1) can be written in operator theoretic form as

$$u + ANu = 0, \tag{2.0.1}$$

where without loss of generality, we have taken $h \equiv 0$.

2.1 Existence and uniqueness theorem

We present the proof of the following theorem which is the main theorem of this chapter.

Theorem 2.1.1 (Browder-Gupta [4]): *Let X be a real Banach space, X^* its conjugate dual space. Let A be a monotone angle-bounded continuous linear mapping of X into X^* with constant of angle-boundedness $c \geq 0$. Let N be a hemicontinuous (possibly nonlinear) mapping of X^* into X such that for a given constant $k \geq 0$,*

$$\langle v_1 - v_2, Nv_1 - Nv_2 \rangle \geq -k\|v_1 - v_2\|^2_{X^*} \tag{2.1.1}$$

for all v_1 and v_2 in X^*. Suppose finally that there exists a constant R with $k(1+c^2)R < 1$ such that for u in X

$$\langle Au, u \rangle \leq R\|u\|^2_X. \tag{2.1.2}$$

Then there exists exactly one solution w in X^* of the nonlinear equation

$$w + ANw = 0. \tag{2.1.3}$$

We prove theorem (2.1.1), using factorization method that consists of splitting the linear operator A via a Hilbert space H. The resulting equation is then solved by using the results of Minty [5] for monotone operator equations. The following proposition of Browder-Gupta [4] enables one to transform the equation (2.1.3) into an equivalent equation in H.

Proposition 2.1.1 (Browder-Gupta [4]): *Let X be a real Banach space, X^* its dual space, A be a bounded linear mapping of X into X^* which is monotone and angle-bounded. Then there exists a Hilbert space H, a continuous linear mapping S of X into H with S^* injective and a bounded skew-symmetric linear mapping B of H into H such that*

$$A = S^*(I + B)S,$$

and the following two inequalities hold:

(i) $\|B\| \leq c$, *with c the constant of angle-boundedness of A;*

(ii) $\|S\|^2 \leq R$ *if and only if for all u in X,* $\langle A(u), u\rangle \leq R\|u\|_X^2$.

Lemma 2.1.1 (Browder-Gupta [4]): *Let H be a given Hilbert space, B a skew-symmetry bounded linear mapping of H into H. Then the bounded linear mapping $I + B$ is monotone bijective mapping of H onto H. Further, for any u in H we have*

$$\big\langle (I + B)^{-1}(u), u\big\rangle \geq \frac{1}{1 + \|B\|^2}\|u\|_H^2.$$

We also need the following lemmas for the proof of theorem (2.1.1).

Lemma 2.1.2 *Let X and Y be Banach spaces and let $T : (X, s) \to (Y, s)$ be a linear continuous map. Then $T : (X, \omega) \to (Y, \omega)$ is continuous, and conversely, where s denotes strong topology and ω denotes weak topology.*

For the proof of lemma (2.1.2), one can see Chidume [1].

Lemma 2.1.3 *Let H be a Hilbert space, X denote a Banach space and X^* its corresponding dual. Suppose that S is a continuous linear mapping of X into H with S^* being its adjoint and N is a hemicontinuous (possibly nonlinear) mapping of X^* into X. Then, the mapping SNS^* of H into H is hemicontinuous.*

Proof.
Let $S : X \to H$ be a linear and continuous map, $N : X^* \to X$ be hemicontinuous. Then by theorem (1.2.1))(a), $S^* : H \to X^*$ is continuous. We show that $SNS^* : H \to H$ is hemicontinuous. To do this, let $u, v \in H$ and let $u_n := (u + t_n v)$ where $(t_n)_{n\geq 1}$ is a sequence of positive numbers such that $t_n \to 0^+$. Then $(u + t_n v) \to u$ as $n \to \infty$. By the continuity of S^*, we have

$$S^*(u_n) = S^*(u + t_n v) \to S^*(u) \in X^*$$

as $n \to \infty$. Let $D := D(N) \subseteq X^*$ denotes the domain of N. Since $S^*(u),\ S^*(v) \in X^*$ and $(S^*(u) + t_n S^*(v)) \in X^*\ \forall\ n \geq 1$ with $t_n \to 0^+$ as $n \to \infty$, by the hemicontinuity of N, we have $N(S^*(u)+t_nS^*(v)) \rightharpoonup N(S^*(u))$ as $n \to \infty$. By the linearity of S^*, we have $N(S^*(u+t_nv)) \rightharpoonup N(S^*(u))$. Thus

$$NS^*(u+t_nv) \rightharpoonup NS^*(u) \in X.$$

Since $S : X \to H$ is linear and continuous from strong topology on X to the strong topology on H, then by the lemma (2.1.2) it is continuous from weak topology on X to the weak topology on H. So by the continuity of S with respect to the weak topology on X, we have, $S(NS^*(u+t_nv)) \rightharpoonup S(NS^*(u))$ i.e $SNS^*(u+t_nv) \rightharpoonup SNS^*(u)$. Therefore, for each $u, v \in H$, $(t_n)_{n\geq 1} \subseteq \mathbb{R} : t_n \to 0^+$ we have $SNS^*(u+t_nv) \rightharpoonup SNS^*(u)$. Hence, SNS^* is hemicontinuous.

2.2 Result of Minty [5]

We give a result of Minty [5] which we shall apply in the proof of theorem (2.1.1).

X is a reflexive Banach space and X^* its 'conjugate' or 'adjoint' space. For $B \subset X$, the symbol $\bar{co}(B)$ denotes the closed convex hull of B.

Definition 2.2.1 *The set B is said to surround x_0 provided $\forall\ z \in X, z \neq 0\ \exists\ t \in (-\infty,\ 0),\ s \in (0, +\infty)$ such that $(x_0 + tz),\ (x_0 + sz)\ \in B$.*

Definition 2.2.2 *The set B is said to surround x_0 densely provided $\forall\ z \in X$, $z \neq 0,\ \forall\ n \in \mathbb{N}\ \exists\ (t_n)_{n\geq1} \subset (-\infty,\ 0),\ (s_n)_{n\geq1} \subset (0, +\infty)$ such that $t_n \to 0,\ s_n \to 0$ and $(x_0 + t_nz),\ \ (x_0 + s_nz)\ \in B$.*

Example 2.2.1 *$B = (-x, x)\,, x \in (0, \infty)$ fixed, contains 0 and surrounds 0 densely.*

Proof.
Let $z \in \mathbb{R},\ z \neq 0.\ \ \forall\ n \geq 1$, take $(t_n)_{n\geq1} = \frac{-x}{\|z\|+n},\ \ (s_n)_{n\geq1} = \frac{x}{\|z\|+n}$.

Clearly $t_n \to 0,\ \ s_n \to 0$ as $n \to +\infty$. Let $p_n := -\frac{xz}{\|z\|+n}$ and $q_n := \frac{xz}{\|z\|+n}$.

Then $p_n \in B \Leftrightarrow |p_n| = |-\frac{xz}{\|z\|+n}| < x$ and $q_n \in B \Leftrightarrow |q_n| = |\frac{xz}{\|z\|+n}| < x$.

Hence B surrounds 0 densely.

Example 2.2.2 *$B = \{x_0\}$ contains $x_0 \in X$ but does not surround x_0. Clearly, $\forall\ \ z \in X,\ z \neq 0,\ \ \forall\ \ t \in (-\infty,\ 0),\ \ s \in (0,\ \infty)$ we have that $(x_0 + tz),\ \ (x_0 + sz)\ \notin B$ because B is singleton. Hence B does not surround x_0.*

Example 2.2.3 $B = [1, 5)$ *contains 1 but does not surround 1.*
Indeed, take $z = 2$ *and any* $t < 0$. *We have* $1 + 2t \notin B$ *as* $1 + 2t < 1$.

Example 2.2.4 $B = [-x, x], x \in (0, \infty)$ *fixed, contains* 0 *and surrounds* 0 *densely.*

Example 2.2.5 $\mathbb{R} \setminus \{0\}$ *does not contain* 0 *but surrounds* 0 *densely.*

Example 2.2.6 *Let dim* $X \geq 2$. *A disc* $D_X(x_0,\ r) = \{x \in X : \ \|x - x_0\| < r\}$, $x_0 \in X$ *fixed,* $r \in \mathbb{R}^+$ *contains* x_0 *and surrounds* x_0 *densely.*

Proof.
Let $z \in X, \ \ z \neq 0$. $\ \forall \ \ n \in \mathbb{N}$ we show that $\exists \ \ (t_n)_{n \geq 1} \subset (-\infty, 0)$, $(s_n)_{n \geq 1} \subset (0, +\infty)$ such that $(x_0 + t_n z), \ \ (x_0 + s_n z) \ \ \in D_X$. Take $t_n = \frac{-r}{\|z\|+n}, \ \ s_n = \frac{r}{\|z\|+n}$. Clearly $t_n \to 0, \ \ s_n \to 0$ as $n \to +\infty$.

Let $p_n := x_0 - \frac{rz}{\|z\|+n}$ and $q_n := x_0 + \frac{rz}{\|z\|+n}$.

Then $p_n \in S_X \ \Leftrightarrow \ \|p_n - x_0\| = \|\frac{-rz}{\|z\|+n}\| = r\|\frac{-z}{\|z\|+n}\| < r$

and $q_n \in S_X \ \Leftrightarrow \ \ \|q_n - x_0\| = \|\frac{rz}{\|z\|+n}\| = r\|\frac{z}{\|z\|+n}\| < r$.

Hence D_X surrounds x_0 densely.

Example 2.2.7 *Let dim* $X \geq 2$. *A sphere* $S_X(x_0, r) = \{x \in X : \ \|x - x_0\| = r\}$, $x_0 \in X$ *fixed,* $r \in \mathbb{R}^+$ *does not contains* x_0 *but surrounds* x_0.

Proof.
Let $z \in X, \ \ z \neq 0$. We show that there exists $t \in (-\infty, \ 0), \ s \in (0, +\infty)$ such that $(x_0 + tz), \ \ (x_0 + sz) \ \in S_X$.
Let $t = \frac{-r}{\|z\|}, \ \ s = \frac{r}{\|z\|}, \ \ p := x_0 - \frac{rz}{\|z\|}$ and $q := x_0 + \frac{rz}{\|z\|}$.
Then $p \in S_X \Leftrightarrow \|p - x_0\| = \|\frac{-rz}{\|z\|}\| = r$ and $q \in S_X \Leftrightarrow \ \|q - x_0\| = \|\frac{rz}{\|z\|}\| = r$.
Hence S_X surrounds x_0.

Example 2.2.8 *Let dim* $X \geq 2$. *and* $B_X(x_0, r) = \{x \in X : \|x - x_0\| \leq r\}$, $x_0 \in X$ *fixed,* $r \in \mathbb{R}^+$. B *is a ball with centre* x_0, *radius* r *and it contains the boundary but does not surround the boundary.*

Theorem 2.2.1 (Minty [5]) *Let* $D \subset X$ *be bounded and surrounds* 0*; let* $E \subset X$ *contain* $\bar{co}(D)$ *and surround every point of* $\bar{co}(D)$ *densely. Let*

$$f : E \to X^*$$

be monotone and hemicontinuous at every point of $\bar{co}(D)$*, and suppose*

$$u \in D \ \textit{implies} \ \ \langle u, f(u) \rangle \geq 0. \tag{2.2.1}$$

Then there exists $u \in \bar{co}(D)$ *such that* $f(u) = 0$.

In a typical application, D would be the boundary of a large ball with centre 0, and E would be an open sphere containing D.

Remark 2.2.1 *$(I+B)^{-1}$ is a continuous linear map of H into H. Indeed,*

(i) $(I+B) \in B(H,H)$;

(ii) $(I+B)$ is bijective.

Thus, by the open mapping theorem, the map $(I+B)^{-1}$ is continuous.

2.3 Proof of theorem (2.1.1)

In this section, we give the Proof of Theorem (2.1.1).

Proof.
Suppose $w \in X^*$ is a solution of the equation (2.1.3). By proposition (2.1.1), $A = S(I+B)S^*$ and equation (2.1.3) becomes

$$w + S^*(I+B)SNw = 0. \tag{2.3.1}$$

By linearity of S^*, we have that

$$w = -S^*(I+B)SNw = S^*\left(-(I+B)SNw\right)$$

which implies that w is in the range of S^*. Since S^* is injective, there is a unique u in H such that $w = S^*(u)$ and therefore equation (2.3.1) becomes

$$S^*u + S^*(I+B)SNS^*u = 0, \tag{2.3.2}$$

i.e.,

$$S^*(u + (I+B)SNS^*u) = 0. \tag{2.3.3}$$

By linearity of S^*, equation (2.3.3) is equivalent to

$$S^*(u + (I+B)SNS^*u) = S^*(0). \tag{2.3.4}$$

Since S^* is injective, equation (2.3.4) is equivalent to

$$u + (I+B)SNS^*u = 0. \tag{2.3.5}$$

Thus, solving for a unique $w \in X^*$ that satisfies equation (2.1.3) is equivalent to solving for a unique $u \in H$ that satisfies equation (2.3.5).

Conversely, suppose that there exists a unique $u \in H$ that satisfies equation (2.3.5). Then $S^*(u + (I+B)SNS^*u) = S^*(0)$ which by linearity of S^* is equivalent to

$$S^*u + S^*(I+B)SNS^*u) = 0.$$

which is equivalent to

$$w + ANw = 0;$$

since $w = S^*(u)$ and $A = S^*(I+B)S$. Therefore, solving for a unique $u \in H$ that satisfies equation (2.3.5) is equivalent to solving for a unique $w \in X^*$ that satisfies equation (2.1.3). Hence equation (2.1.3) has exactly one solution in X^* if and only if equation (2.3.5) has exactly one solution in H. Now, by lemma (2.1.1), equation (2.3.5) is equivalent to the equation

$$(I+B)^{-1}(u) + SNS^*(u) = 0. \tag{2.3.6}$$

We now apply the result of Minty [5]. Define $f = (I+B)^{-1} + SNS^*$. H is reflexive. Take $X = H$, and $D = S_H(0,r) = \{u \in H : \|u\| \leq r\}, r \in \mathbb{R}^+$. Clearly, D is bounded and surrounds 0. $\bar{co}(D) = D$ since D is closed and convex. Take $E = H$ and define $f : H \to H$. $\forall\ u, v \in H$,

$$\langle f(u) - f(v), u - v\rangle = \big\langle (I+B)^{-1}(u-v), u-v\big\rangle + \langle SNS^*(u) - SNS^*(v), u-v\rangle .$$

Therefore,

$$\begin{aligned} \big\langle (I+B)^{-1}(u-v), u-v\big\rangle &\geq \frac{1}{1+\|B\|^2}\|u-v\|_H^2 \quad \text{(by lemma (2.1.1))} \\ &\geq \frac{1}{1+c^2}\|u-v\|_H^2 \quad \text{(by proposition (2.1.1)(i)).} \end{aligned}$$

Thus

$$\big\langle (I+B)^{-1}(u-v), u-v\big\rangle \geq \frac{1}{1+c^2}\|u-v\|_H^2 \tag{2.3.7}$$

On the other hand,

$$\begin{aligned} \langle SNS^*(u) - SNS^*(v), u-v\rangle &= \langle S^*(u) - S^*(v), NS^*(u) - NS^*(v)\rangle \\ &\geq -k\|S^*(u) - S^*(v)\|_{X^*}^2 \quad \text{(hypothesis of theorem (2.1.1))} \\ &\geq -k\|S^*(u-v)\|_{X^*}^2 \quad \text{(by linearity of } S^*) \\ &\geq -kR\|u-v\|_H^2 \quad \text{(by proposition (2.1.1)(ii)).} \end{aligned}$$

Thus

$$\langle SNS^*(u) - SNS^*(v), u-v\rangle \geq -kR\|u-v\|_H^2 \tag{2.3.8}$$

Combining inequalities ((2.3.7) and (2.3.8)), we have that

$$\langle f(u) - f(v),\ u-v\rangle \geq (\frac{1}{1+c^2} - kR)\|u-v\|_H^2 = c_1\|u-v\|_H^2 \tag{2.3.9}$$

where $c_1 = \frac{1}{1+c^2} - kR > 0$ since $k(1+c^2)R < 1$ by hypothesis of theorem (2.1.1).
Thus f is a monotone mapping of H into H.
Suppose that $u \neq v$. Then by equation (2.3.9) we have

$$\langle f(u) - f(v), u-v\rangle > 0 \Rightarrow f(u) \neq f(v).$$

Thus f maps H into H injectively.
Next, we show that f is hemicontinuous. Let $u, v \in H$ and $(t_n) \subset \mathbb{R}^*$ such that

$t_n \to 0^+$. We show that $f(u+t_nv) \rightharpoonup f(u)$ as $n \to \infty$ because H is reflexive. Let $w \in H$.

$$\begin{aligned}
\langle f(u+t_nv) - f(u),\ w\rangle &= \langle((I+B)^{-1} + SNS^*)(u+t_nv) \\
&\quad -((I+B)^{-1} + SNS^*)(u),\ w\rangle \\
&= \langle(I+B)^{-1}(u+t_nv) - (I+B)^{-1}(u) \\
&\quad +SNS^*(u+t_nv) - SNS^*(u),\ w\rangle \\
&= \left\langle(I+B)^{-1}(u+t_nv) - (I+B)^{-1}(u),\ w\right\rangle \\
&\quad + \langle SNS^*(u+t_nv) - SNS^*(u),\ w\rangle \\
&= \left\langle(I+B)^{-1}(u+t_nv-u),\ w\right\rangle \\
&\quad + \langle SNS^*(u+t_nv) - SNS^*(u),\ w\rangle \\
&= t_n\left\langle(I+B)^{-1}(v), w\right\rangle \\
&\quad + \langle SNS^*(u+t_nv) - SNS^*(u),\ w\rangle \to 0
\end{aligned}$$

since $t_n \to 0^+$ and SNS^* is hemicontinuous by lemma (2.1.3). Thus, f is hemicontinuous.

Moreover, $\forall\ u \in D$, we have $\langle f(u),\ u\rangle \geq 0$.

It then follows from the result of Minty [5] that there exists $u \in D$ such that $f(u) = 0$. Furthermore, since f is injective, then u is unique. Hence equation(2.3.5) has exactly one solution in H and so by the preceding discussion, equation(2.1.3) has exactly one solution in X^*.

CHAPTER 3

Existence and Uniqueness Results Using Variational Methods

Let X be a real Banach space with X^* its corresponding conjugate. The problem of solving $w + ANw = 0$ in X^* is transformed into the problem of solving a suitable equation

$$Tu = 0 \qquad (u \in H) \tag{3.0.1}$$

in a Hilbert space with T a potential operator in H such that its potential has a local minimum which is then used as a solution of equation (3.0.1). This is a consequence of the Euler's theorem which will be stated shortly.

3.1 Gâteaux derivative and gradient

Let X and Y be two real normed spaces.

Definition 3.1.1 (Interior point): *A point $u \in A$ is called an interior point of A provided that there exists $\epsilon > 0$ such that $B(u, \epsilon) \subseteq A$. The set of all interior points of A is called the interior of A, denoted $\mathring{A}$.*

Definition 3.1.2 (Closure): *The closure of A, $\bar{A}$ is the smallest closed set containing A.*

Definition 3.1.3 (Boundary): *The boundary of $A, \partial A$ is defined as $\partial A := \bar{A} \cap \bar{A}^c$.*

Definition 3.1.4 (Gâteaux Derivative): *Let $f : U \subset X \to Y$ be a map with U open and nonempty. The function f is said to have a Gâteaux derivative* *at $u \in U$ if there exists a bounded linear map of X into Y denoted by $D_G f(u)$ such that for each h in X we have*

$$\lim_{t\to 0}\frac{f(u+th)-fu}{t} = \langle D_G f(u), h\rangle . \tag{3.1.1}$$

We say that f is Gâteaux differentiable if it has a Gâteaux derivative at each u in U. We shall use $f'(u)$ to mean $D_G f(u)$.

Example 3.1.1 *The function $f:\mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = x^2+y^2$ is Gâteaux differentiable at every point $(x_0,y_0) \in \mathbb{R}^2$. Indeed, let $u_0 = (x_0,y_0)$ and $h = (h_1,h_2) \in \mathbb{R}^2\setminus\{0\}$. Then*

$$\begin{aligned} f(u_0+th)-f(u_0) &= (x_0+th_1)^2+(y_0+th_2)^2-(x_0^2+y_0^2)\\ &= x_0^2+2x_0th_1+t^2h_1^2+y_0^2+2ty_0h_2+t^2h_2^2-(x_0^2+y_0^2)\\ &= 2t(x_0h_1+y_0h_2)+t^2(h_1^2+h_2^2) \quad \forall \ \ t\in\mathbb{R}. \end{aligned}$$

It follows that $\lim\limits_{t\to 0}\frac{f(u_0+th)-f(u_0)}{t} = 2(x_0h_1+y_0h_2) = 2\langle u_0,\ h\rangle$.
Since the map $h \mapsto 2\langle u_0,\ h\rangle$ is linear and continuous from $\mathbb{R}^2$ to $\mathbb{R}$, we conclude that f is Gâteaux differentiable and $f'(u_0)h = 2\langle u_0,\ h\rangle \quad \forall \ \ h\in\mathbb{R}^2$.

Example 3.1.2 *Let f be a functional defined from $\mathbb{R}^2$ into $\mathbb{R}$ by:*

$$f(x_1,x_2) = \begin{cases} \frac{x_1x_2^4}{x_1^2+x_2^8}, & \text{if } x_1 \neq 0\\ 0, & otherwise. \end{cases}$$

The functional f is Gâteaux differentiable at $(0,0)$.

Indeed, let $h = (h_1,h_2) \in \mathbb{R}^2\setminus\{0\}$ and $t \neq 0$, we have:

$$\frac{f(th)-f(0)}{t} = \begin{cases} \frac{t^2h_1h_2^4}{h_1^2+t^6h_2^8}, & \text{if } h_1 \neq 0\\ 0, & otherwise. \end{cases}$$

This implies that $\lim\limits_{t\to 0}\frac{f(tx)-f(0)}{t} = 0$. *Hence f is Gâteaux differentiable at $(0,0)$ and $f'(0)\equiv 0$.*

Definition 3.1.5 (Potential): *A mapping G of X into X^* is said to be potential (weakly potential) if it is a Gâteaux derivative of some Gâteaux differentiable function. That is, a mapping G of X into X^* is said to be potential (weakly potential) if there exists a functional f of X into $\mathbb{R}$ such that for all u and v in X we have*

$$\lim_{t\to 0}\frac{f(u+tv)-f(u)}{t} = \langle Gu, v\rangle . \tag{3.1.2}$$

The functional f is called the potential of G and G is said to be the gradient of f, written $grad(f) = G$.

Example 3.1.3 *Define a functional f by $f(u) = \frac{1}{2}\langle u, u\rangle$ for $u \in H$, where H is a real Hilbert space. For all $u, h \in H$, and $t \in \mathbb{R}$ we have:*

$$\begin{aligned}\frac{1}{2}\langle u+th, u+th\rangle &= \frac{1}{2}\langle u, u\rangle + \frac{1}{2}t\langle u, h\rangle + \frac{1}{2}t\langle h, u\rangle + \frac{1}{2}t^2\langle h, h\rangle \\ &= \frac{1}{2}\langle u, u\rangle + t\langle u, h\rangle + \frac{1}{2}t^2\langle h, h\rangle .\end{aligned}$$

Therefore $\frac{1}{2}\langle u+th, u+th\rangle - \frac{1}{2}\langle u, u\rangle = t\langle u, h\rangle + \frac{1}{2}t^2\langle h, h\rangle$. *Thus*

$$\lim_{t\to 0}\frac{1}{2}\frac{\langle u+th,\ u+th\rangle - \langle u,\ u\rangle}{t} = \lim_{t\to 0}(\langle u, h\rangle + \frac{1}{2}t\langle h, h\rangle) = \langle u, h\rangle .$$

Hence, $grad(f) = I$.

3.2 Maxima and minima of functions

Let X be a real normed linear space. We consider a real functional $f : X \to \mathbb{R}$. For simplicity, it is assumed that f is defined for all values of u in X.

Definition 3.2.1 (Local minimum): *A functional f is said to have a local minimum at $u = u_0$ if for some positive ϵ, $f(u) - f(u_0) \geq 0$ for all $u \in B(u_0, \epsilon)$.*

Definition 3.2.2 (Local maximum): *A functional f is said to have a local maximum at $u = u_0$ if for some positive ϵ, $f(u) - f(u_0) \leq 0$ for all $u \in B(u_0, \epsilon)$.*

A common name for a maximum or a minimum is an **extremum**.

Remark 3.2.1 *When G is potential, then $Gu = 0$ whenever u is a local minimum or maximum of f, where $grad(f) = G$.*

Definition 3.2.3 (Stationary point): *This is the point at which the derivative of a differentiable function f vanishes and f is said to be stationary at u_0 whenever $f'(u_0) = 0$.*

Theorem 3.2.1 *Let u_0 be a stationary point of f with a continuous second derivative. Then $f(u_0)$ is a maximum for $f''(u_0) < 0$ and a minimum for $f''(u_0) > 0$.*

For the proof of theorem (3.2.1), see, for instance, Lauwerier [7].

Examples

(a) $f(u, v) = u^2 + v^2 + 1$ has minimum at $(0, 0)$.

(b) $f(u, v) = u^4 + v^4 + 1$ has minimum at $(0, 0)$.

Theorem 3.2.2 (Euler's Theorem): *Let X be a real normed linear space and U a subset of X with nonempty interior. Let $f : U \to \mathbb{R}$ be a functional and suppose that $u \in \overset{\circ}{U}$ is a local extremum of f. If f is Gâteaux differentiable at u, then $f'(u) = 0$.*

Proof.
Let $u_o \in \overset{\circ}{U}$ be an extremum at which f is Gâteaux differentiable. We assume, without loss of generality, that u_0 is a local minimum. So let $\epsilon > 0$ such that $B(u_0, \epsilon) \subset U$ and $f(u_0) \leq f(u)$ for every $u \in B(u_0, \epsilon)$. Then for any $h \in X \setminus \{0\}$, setting $\delta_h := \frac{\epsilon}{\|h\|}$, we obtain that for any $t \in \mathbb{R}$ with $|t| < \delta_h$,

$$\|u_0 + th - u_0\| < \epsilon.$$

So

$$\frac{f(u_0 + th) - f(u_0)}{t} \geq 0, \quad t \in (0, \delta_h) \tag{3.2.1}$$

and

$$\frac{f(u_0 + th) - f(u_0)}{t} \leq 0, \quad t \in (-\delta_h, 0). \tag{3.2.2}$$

Taking limit as $t \to 0$, in (3.2.1) and (3.2.2) we get

$$\langle f'(u_0), h \rangle \geq 0 \tag{3.2.3}$$

and

$$\langle f'(u_0), h \rangle \leq 0. \tag{3.2.4}$$

Hence $\langle f'(u_0), h \rangle = 0$. Since h was arbitrary, $f'(u_0) = 0$. The proof is complete.

3.3 Fundamental theorems of optimization

We take the following to be the definition of the terms which are used in this chapter.

Definition 3.3.1 (Lower semi-continuity): *Let X be a real Banach space. A functional f defined on X is said to be lower semi-continuous provided that $x_n \to x \Rightarrow f(x) \leq \liminf_n f(x_n)$.*

Definition 3.3.2 (Weak (sequential) lower semi-continuity): *Let X be a real Banach space. A functional f defined on X is said to be weakly lower semi-continuous at $x \in X$ if $\forall$ $(x_n)_{n \geq 1} \subset X$, we have $x_n \rightharpoonup x \Rightarrow f(x) \leq \liminf_n f(x_n)$.*

Definition 3.3.3 *The epigraph of f is the set defined by*

$$epi(f) := \{(x, \ \alpha) \in X \times \mathbb{R} : \ x \in D(f) \ \text{ and } \ f(x) \leq \alpha\},$$

where $D(f)$ is the domain of f.

Proposition 3.3.1 *Let $f : X \to \mathbb{R}$ be any map. Then f is convex and lower semi-continuous $\Leftrightarrow$ f is convex and weakly lower semi-continuous.*

Proof.
f is convex and lower semi-continuous
$\Leftrightarrow$ $epi(f)$ is convex and closed
$\Leftrightarrow$ $epi(f)$ is convex and weakly closed
$\Leftrightarrow$ f is convex and weakly lower semi-continuous.

Theorem 3.3.1 *Let X be a real reflexive Banach space and let K be a closed convex bounded and nonempty subset of X. Let $f : X \to \mathbb{R}$ be lower semi-continuous and convex. Then there exists $u_0 \in K$ such that $f(u_0) \leq f(u)$ $\forall$ $u \in K$, i.e, $f(u_0) = \inf_{u \in K} f(u)$.*

Theorem 3.3.2 *Let X be a real reflexive Banach space and $f : X \to \mathbb{R}$ be a convex lower semi-continuous functional. Suppose $\lim_{\|u\| \to \infty} f(u) = +\infty$. Then, there exists $u_0 \in X$ such that $f(u_0) \leq f(u)$, $u \in X$, i.e,$f(u_0) = \inf_{u \in X} f(u)$.*

For the proof of the above theorems ((3.3.1) and (3.3.2)), see, for instance, Chidume [1].

Theorem 3.3.3 (Eberlein-Smul'yan:) *A Banach space X is reflexive if and only if every (norm) bounded sequence in X has a subsequence which converges weakly to an element of X.*

For the proof of the theorems (3.3.3), see, for instance, Brezis [8].

3.4 Extension of Vainberg's result to real Banach spaces

A special case of Theorem (2.1.1) is the following.

Theorem 3.4.1 (Browder-Gupta [4]): *Let X be a real Banach space, X^* its conjugate space, A a bounded linear mapping of X into X^* which is monotone and symmetric. Suppose that N is a hemicontinuous (possibly nonlinear) mapping of X^* into X such that for a given $k \geq 0$ and all v_1, v_2 in X^*,*

$$\langle v_1 - v_2, Nv_1 - Nv_2 \rangle \geq -k\|v_1 - v_2\|^2_{X^*}.$$

Suppose that $k\|A\| < 1$. Then the equation

$$w + ANw = 0 \tag{3.4.1}$$

has exactly one solution w in X^*.

The result of Theorem (3.4.1) was obtained by Golomb [9] for $X = L^2(\Omega)$ and by Vainberg [10] for $X = L^p(\Omega)$, using variational methods. Using the result of proposition (2.1.1) and lemma (2.1.1), in this chapter we consider the extension of Vainberg's result [10] to real Banach spaces under the assumption that $A : X \to X^*$ is a linear monotone and symmetric mapping and thus, angle-bounded with constant of angle-boundedness $c = 0$ while $N : X^* \to X$ is a potential mapping satisfying suitable growth conditions [13].

We shall need the following corollary from the proposition (2.1.1) in the proof of the subsequent theorems.

Corollary 3.4.1 $\|S^*\|^2 \leq \|A\|$.

Proof.
Since A is a bounded linear mapping of X into X^*, there exists $R \geq 0$ such that

$$\langle Au, u\rangle \leq R\|u\|_X^2 \;\forall\; u \in X.$$

In particular, take $R = \|A\|$. Also, by the symmetry of A, $A = S^*S$. Therefore for all u in X we have

$$\|Su\|_H^2 = \langle Su, Su\rangle = \langle S^*Su, u\rangle = \langle Au, u\rangle \leq R\|u\|_X^2 = \|A\|.\|u\|_X^2.$$

Thus $\|S\| \leq \sqrt{\|A\|}$. Hence $\|S^*\|^2 \leq \|A\|$ since $\|S\| = \|S^*\|$.

In what follows, $B(0, r)$ denotes an open ball while $\bar{B}$(0,r) denotes its closure and ∂B(0,r) denotes its boundary with centre 0 and radius r. Also, we shall make use of the following known fact.

Proposition 3.4.1 (Petryshyn-Fitzpatrick [13]): *Let X be a reflexive Banach space (in particular, a Hilbert space). Let $f : \bar{B}(0, r) \subseteq X \to \mathbb{R}$ be a weakly semi-continuous functional. Then f assumes its infimum on $\bar{B}(0, r)$. Furthermore, if $f(u) > f(0)$ for all $u \in \partial B(0, r)$, then f attains a local minimum at an interior point of $\bar{B}(0, r)$.*

Proof.
Let $\alpha = \inf_{u\in\bar{B}(0,r)} f(u)$. This is implies that $\alpha \leq f(u) \;\forall\; u \in \bar{B}(0, r)$ and there exists a sequence $(u_n)_{n\geq1} \subseteq \bar{B}(0, r)$ such that $\lim_n f(u_n) = \alpha$. Since $(u_n)_{n\geq1} \subseteq \bar{B}(0, r)$, $(u_n)_{n\geq1}$ is bounded. Eberlein-Smul'yan theorem implies that there exists $(u_{n_j})_{j\geq1} \subset (u_u)_{n\geq1}$ such that $u_{n_j} \rightharpoonup u^*$. $\bar{B}(0, r)$ closed and convex implies that it is weakly closed. Thus $u^* \in \bar{B}(0, r)$. f is weakly lower semi-continuous implies that

$$f(u^*) \leq \liminf_j f(u_{n_j}). \tag{3.4.2}$$

Since $\{f(u_n)\}$ converges to α, it follows that

$$f(u^*) \leq \lim_j f(u_{n_j}) = \lim_n f(u_n) = \alpha. \tag{3.4.3}$$

Thus, f assumes its infimum on $\bar{B}(0,r)$.

Furthermore, if $f(u) > f(0)$ for all $u \in \partial B(0,r)$, then $f(u^*) \leq f(0) < f(u)$ for all $u \in \partial B(0,r)$. So $u^* \notin \partial B(0,r)$. Thus f attains a local minimum at an interior point of $\bar{B}(0,r)$.

Theorem 3.4.2 (Petryshyn-Fitzpatrick [13]): *Let X be a real reflexive Banach space and let A be a linear , monotone and symmetric mapping of X into X^*. Suppose f is a weakly (sequential) lower semicontinuous functional on X^* such that*

$$f(u) \geq -\frac{1}{2}a_1\|u\|^2 - a_2\|u\|^\delta - a_3 \tag{3.4.4}$$

where $a_1\|A\| < 1, a_2 > 0, a_3 > 0$ and $0 < \delta < 2$. Suppose also that $N : X^* \to X$ is such that $grad(f) = N$. Then the equation (3.4.1) has a solution in X^*.

Proof.
From proposition (2.1.1)(i), when A is symmetric, $B = 0$. Therefore, in terms of proposition (2.1.1), it suffices to find a solution of the equation

$$u + SNS^*u = 0, \quad u \in H. \tag{3.4.5}$$

Define a functional by $q(u) = \frac{1}{2}\langle u, u\rangle + f(S^*u)$ for $u \in H$.
We note that q is weakly lower semicontinuous. Indeed, suppose $u_n \rightharpoonup u$ in H. Then

$\frac{1}{2}\langle u, u\rangle = \frac{1}{2}\|u\|^2 \leq \liminf_n \frac{1}{2}\|u_n\|^2$ by continuity and convexity of $\|.\|^2$.

and $S^*(u_n) \rightharpoonup S^*(u)$ in X^* by continuity of S^* and lemma (2.1.2).

Thus $f(S^*u) \leq \liminf_n f(S^*u_n)$ by the weakly lower semicontinuity of f.
Consequently,

$$\begin{aligned} q(u) &= \frac{1}{2}\langle u, u\rangle + f(S^*u) \leq \liminf_n \frac{1}{2}\langle u_n, u_n\rangle + \liminf_n f(S^*u_n) \\ &\leq \liminf_n \left\{\frac{1}{2}\langle u_n, u_n\rangle + f(S^*u_n)\right\} \quad (\text{ by subadditivity of } \liminf) \\ &= \liminf_n q(u_n). \end{aligned}$$

Also, using proposition (3.4.1) and corollary (3.4.1), for each u in H, we have

$$\begin{aligned}
q(u) &= \frac{1}{2}\|u\|^2 + f(S^*u)\\
&\geq \frac{1}{2}\|u\|^2 - \frac{1}{2}a_1\|S^*u\|^2 - a_2\|S^*u\|^\delta - a_3 \quad \text{(by the equation (3.4.4))}\\
&\geq \frac{1}{2}\|u\|^2 - \frac{1}{2}a_1\|S^*\|^2.\|u\|^2 - a_2\|S^*\|^\delta.\|u\|^\delta - a_3 \;\; (S^* \text{ bounded })\\
&\geq \frac{1}{2}\|u\|^2 - \frac{1}{2}a_1\|A\|.\|u\|^2 - a_2\|A\|^{\frac{\delta}{2}}.\|u\|^\delta - a_3 \;\; \text{(by corollary (3.4.1))}\\
&= \frac{1}{2}(1 - a_1\|A\|)\|u\|^2 - a_2\|A\|^{\frac{\delta}{2}}.\|u\|^\delta - a_3\\
&\geq c_1\|u\|^2 - \frac{c_1}{2}\|u\|^2 - a_3 \;\; \text{provided } \;\|u\| > \left(\frac{2c_2}{c_1}\right)^{\frac{1}{2-\delta}}\\
&= \frac{c_1}{2}\|u\|^2 - a_3
\end{aligned}$$

where $c_1 = \frac{1}{2}(1 - a_1\|A\|)$ and $c_2 = a_2\|A\|^{\frac{\delta}{2}}$. Hence, we see from our conditions on $a_1,\ a_2,\ a_3$ and δ that $q(u) \to \infty$ as $\|u\| \to \infty$.

Consequently, there exists $r > 0$ such that $q(u) > q(0)$ for all $u \in \partial B(0, r) \subset H$. Thus, by proposition (3.4.1), q attains a local minimum at an interior point of $\bar{B}(0, r)$.

Next, we want to evaluate $grad(q)$.
Observe that for all $u,\ h\ \in H$

$$q(u) = \tfrac{1}{2}\langle u, u\rangle + f(S^*u) \text{ and;}$$

$$\begin{aligned}
q(u+th) &= \tfrac{1}{2}\langle u+th,\ u+th\rangle + f(S^*(u+th)), \quad \forall\ \ t \in \mathbb{R}\\
&= \tfrac{1}{2}\langle u+th,\ u+th\rangle + f(S^*u + tS^*h)\text{: by linearity of } S^*.
\end{aligned}$$

Since $grad(\frac{1}{2}\langle u,\ u\rangle) = I$ (see example (3.1.3)) and grad(f)=N, we have

$$\begin{aligned}
Dq(u,h) &= \lim_{t\to 0}\tfrac{q(u+th)-q(u)}{t}\\
&= \lim_{t\to 0}\tfrac{1}{2}\tfrac{\langle u+th,\ u+th\rangle - \langle u,u\rangle}{t} + \lim_{t\to 0}\tfrac{f(S^*u+tS^*h)-f(S^*u)}{t}\\
&= \langle Iu, h\rangle + (NS^*u, S^*h)\\
&= \langle Iu, h\rangle + \langle SNS^*u, h\rangle\\
&= \langle Iu + SNS^*u,\ h\rangle\\
&= \langle (I + SNS^*)u,\ h\rangle\,.
\end{aligned}$$

Therefore $grad(q) = I + SNS^*$.
Thus, the mapping $I + SNS^* : H \longrightarrow H$ has a zero, i.e, equation (3.4.5) is

solvable. Hence equation (3.4.1) is solvable. Moreover, $\forall \ u, \ v \ \in H$,

$$\begin{aligned}\langle (I+SNS^*)(u)-(I+SNS^*)(v), u-v\rangle &= \langle I(u-v), u-v\rangle \\ &\quad + \langle SNS^*(u)-SNS^*(v), u-v\rangle .\end{aligned}$$

$$\langle I(u-v), u-v\rangle = \langle u-v, u-v\rangle = \|u-v\|_H^2. \tag{3.4.6}$$

For any $u, v \in H$ we have

$$\begin{aligned}\langle SNS^*(u)-SNS^*(v), u-v\rangle &= \langle S^*(u)-S^*(v), NS^*(u)-NS^*(v)\rangle \\ &\geq -k\|S^*(u)-S^*(v)\|_{X^*}^2 \ \text{(by hypothesis of theorem (3.4.1))} \\ &\geq -k\|S^*(u-v)\|_{X^*}^2 \ \text{(by linearity of } S^*) \\ &\geq -k\|S^*\|^2.\|u-v\|_H^2 \ \text{(by boundedness of } S^*) \\ &= -k\|A\|.\|u-v\|_H^2 \ \text{(by corollary (3.4.1)).}\end{aligned}$$

Thus

$$\langle SNS^*(u)-SNS^*(v), u-v\rangle \geq -k\|A\|.\|u-v\|_H^2. \tag{3.4.7}$$

Combining inequalities ((3.4.6) and (3.4.7)), we have that

$$\langle (I+SNS^*)(u)-(I+SNS^*)(v), u-v\rangle \geq (1-k\|A\|)\|u-v\|_H^2 = c_1\|u-v\|_H^2 \tag{3.4.8}$$

where $c_1 = 1-k\|A\| > 0$ since $k\|A\| < 1$ by hypothesis of theorem(3.4.1).

Suppose that $u \neq v$. From equation(3.4.8) we obtain that

$$\langle (I+SNS^*)(u)-(I+SNS^*)(v), u-v\rangle > 0.$$

This implies

$$(I+SNS^*)(u) \neq (I+SNS^*)(v).$$

Thus it follows that $(I+SNS^*)$ maps H into H injectively. Therefore equation (3.4.5) has exactly one solution in H and so by the preceding discussion, equation (3.4.1) has exactly one solution in X^*.

While in Theorem(3.4.2) we assumed a growth condition on the potential of N to obtain the existence of a solution, in the next theorem we will assume, in addition to the potentialness of N, that it has a Gâteaux derivative N' and place a growth condition on N'. The following proposition will be needed.

Proposition 3.4.2 *Vainberg [11] Let X be a reflexive Banach space (in particular, a Hilbert space). Suppose $f : X \to \mathbb{R}$ is such that it has first and second Gâteaux derivatives on all of X, with the latter satisfying the inequality*

$$D^2f(u,h,h) \geq \|h\|\gamma(\|h\|)$$

and $D^2f(tu,h,h)$ being continuous in $t \in [0,\ 1]$ for u and h fixed, where $\gamma(t)$ is a nonnegative continuous function defined for $t \geq 0$ and such that $\lim\limits_{t\to\infty} \gamma(t) = \infty$. Then there exists $u_0 \in X$ such that f has a local minimum at u_0.

Theorem 3.4.3 (Petryshyn-Fitzpatrick [13]): *Let X be a reflexive Banach space with $A : X \to X^*$ linear, monotone and symmetric. Let $N : X^* \to X$ be potential and have a Gateaux derivative which satisfies the inequality*

$$DN(u, v, v) \geq -a\|v\|^2 \qquad (v, u \in X^*)$$

and DN(tu, v, v) is continuous in $t \in [0, 1]$ for u and v fixed, where $a\|A\| < 1$. Then the equation $w + ANw = 0$ has a solution in X^*.

Proof.
Using proposition(2.1.1), it suffices to find a solution in H to

$$u + SNS^*u = 0.$$

Define $q(u) = \frac{1}{2}\langle u, u\rangle + f(S^*u)$ for $u \in H$, where $grad(f) = N$. We have

$$Dq(u, h) = \lim_{t\to 0} \frac{q(u + th) - q(u)}{t} = \langle u, h\rangle + (NS^*u, S^*h) \qquad (3.4.9)$$

and

$$\begin{aligned}
D^2q(u, k, h) &= \lim_{t\to 0}\frac{1}{t}\{Dq(u + tk, h) - Dq(u, h)\} \\
&= \lim_{t\to 0}\frac{1}{t}\{\langle u + tk, h\rangle + \langle NS^*(u + tk), S^*h\rangle - \langle u, h\rangle - \langle NS^*u, S^*h\rangle\} \\
&= \lim_{t\to 0}\frac{1}{t}\{\langle tk, h\rangle + \langle NS^*(u + tk) - NS^*u, S^*h\rangle\} \\
&= \langle k, h\rangle + \lim_{t\to 0}\frac{1}{t}\langle N(S^*u + tS^*k) - NS^*u), S^*h\rangle \\
&= \langle k, h\rangle + DN(S^*(u), S^*(k), S^*(h)).
\end{aligned}$$

Hence by the hypothesis of theorem (3.4.3), we have the inequality

$$\begin{aligned}
D^2q(u, h, h) &= \langle h, h\rangle + DN(S^*(u), S^*(h), S^*(h)) \\
&\geq \|h\|^2 - a\|S^*\|^2.\|h\|^2 \quad \text{by the hypothesis of theorem (3.4.3)} \\
&\geq \|h\|^2 - a\|A\|\|h\|^2 \;\; \text{by corollary (3.4.1)} \\
&= (1 - a\|A\|)\|h\|^2 \\
&= \|h\|\gamma(\|h\|)
\end{aligned}$$

where $\gamma(\|h\|) = (1 - a\|A\|)\|h\|$ and clearly $\gamma(\|h\|) \to \infty$ as $\|h\| \to \infty$ (since $a\|A\| < 1$). We now invoke proposition(3.4.2) to conclude that q has a local minimum. Hence $grad(q)$ has a zero. Thus, there exists $u_0 \in H$ such that $u_0 + SNS^*u_0 = 0$. Thus, there exists $w \in X^*$ such that $w + ANw = 0$. Also, $(I + SNS^*)$ maps H into H injectively. Thus, there exists a unique $u_0 \in H$ such that $u_0 + SNS^*u_0 = 0$. Hence there exists a unique $w \in X^*$ such that $w + ANw = 0$.

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